Received: *February 4, 1999*, Accepted: *March 12, 1999*

On the quantum Batalin-Vilkovisky formalism and the renormalization of non linear symmetries

Glenn Barnich*

Physique Théorique et Mathématique, Université Libre de Bruxelles, Campus Plaine C.P. 231, B-1050 Bruxelles.
E-mail: gbarnich@ulb.ac.be

Abstract: The most convenient tool to study the renormalization of a Lagrangian field theory invariant under non linear local or global symmetries is the proper solution to the master equation of the extended antifield formalism. It is shown that, from the knowledge of the BRST cohomology, it is possible to explicitly construct a further extension of the formalism containing all the observables of the theory and satisfying an extended master equation, with some of the features of the quantum Batalin-Vilkovisky master equation already present at the classical level. This solution has the remarkable property that all its infinitesimal deformations can be extended to complete deformations. The deformed solutions differs from the original one through the addition of terms related to coupling constant and anticanonical field-antifield redefinitions. As a consequence, all theories admitting an invariant regularization scheme are shown to be renormalizable while preserving the symmetries, in the sense that both the subtracted and the effective action satisfy the extended master equation, and this independently of power counting restrictions. The anomalous case is also studied and a suitable definition of the Batalin-Vilkovisky "Delta" operator in the context of dimensional renormalization is proposed.

Keywords: Renormalization Regularization and Renormalons, BRST Quantization, Global Symmetries, Anomalies in Field and String Theories.

*Chargé de Recherches du Fonds National Belge de la Recherche Scientifique.

JHEP03(1999)010

Contents

1. Introduction

The best known example of the renormalization of a theory invariant under a non linear symmetry is probably non abelian Yang-Mills theory: on the level of the gauge fixed Faddeev-Popov action [1], gauge invariance is expressed through invariance

JHEP03(1999)010

under the non linear global BRST symmetry [2, 3]. Some of the crucial points in the analysis are:

(i) the importance of the BRST cohomology as a constraint on the anomalies and the counterterms of the theory,

(ii) the anticanonical structure of the theory in terms of the fields and the sources, to which the BRST variations are coupled, together with the compact reformulation of all the Ward identities in terms of the Zinn-Justin equation [4], and

(iii) the insight that BRST exact counterterms can be absorbed by anticanonical fields and sources redefinitions [5].

The question whether the remaining counterterms can be absorbed by a redefinition of the coupling constants of the theory could be settled to the affirmative in the power counting renormalizable case, through an exhaustive enumeration of all possible renormalizable interactions [2]. In the case where one includes higher dimensional gauge invariant operators, such a property depends crucially on a conjecture by Kluberg-Stern and Zuber [6] on the BRST cohomology in ghost number 0, which states that it should be describable by off-shell gauge invariant operators not involving the ghosts or the sources. This conjecture can be shown [7] to hold in the semi-simple case for which it has been originally formulated, but its generalization in the presence of abelian factors is not valid, and this even for power counting renormalizable theories. In this last case, the counterterms violating the generalized Kluberg-Stern and Zuber conjecture have been shown to be absent by more involved arguments from renormalization theory [8], so that renormalizability still holds, even if the conjecture does not.

The classical Batalin-Vilkovisky formalism [9] (for reviews, see e.g. [10, 11]) extends the above techniques to the case of general gauge theories with open gauge algebras and structure functions, the invariance of the action being expressed through the central master equation. A further extension consists in controlling at the same time the renormalization of non linear global symmetries by coupling them with constant ghosts [12, 13].

A detailed analysis of the compatibility of the renormalization procedure with invariance expressed through the master equation has been performed in [14, 15, 16], where it has been shown that the renormalized action is a deformation of the starting point solution to the master equation. Independently of this result, the fundamental problem of locality of the construction is raised and a locality hypothesis is stated [14]. This hypothesis can be reinterpretated in a more general framework as the assumption that the cohomology of the Koszul-Tate differential [17, 18] vanishes in the space of local functionals. While the assumption holds under certain conditions, which are in particular fulfilled for the construction of the solution of the master equation,

JHEP03(1999)010

thus guaranteeing its locality [18], it does not hold in general; the obstructions are related to the non trivial global currents of the theory [19], and give rise to BRST cohomology classes with a non trivial antifield dependence.

A consequence of this is that there exist observables which cannot be made off-shell gauge invariant, even in the case of closed gauge theories, so that the associated deformed solutions of the master equation cannot be related by a field, antifield and coupling constant renormalization to the starting point solution extended by coupling all possible off-shell observables compatible with the symmetries.

In [20], renormalization in the context of the Batalin-Vilkovisky formalism is reconsidered precisely under the assumption that there are no such deformations, i.e. in the closed case under the analog of the Kluberg-Stern and Zuber conjecture, and in the open case under the conjecture that all the BRST cohomology is already contained in the solution to the master equation coupled with independent coupling constants,[1] with the conclusion, that the infinities can then be absorbed by renormalizations.

Finally, in [21] the problem of renormalization under non linear symmetries is readdressed in the context of effective field theories: it is for instance shown that semi-simple Yang-Mills theory and gravity, to which are coupled all possible (power counting non renormalizable) off-shell observables, are such that all the local counterterms needed to cancel the infinities, can be absorbed through coupling constants, field and antifield renormalizations, while preserving the symmetry (in the form of the Batalin-Vilkovisky master equation). Theories possessing this last property, even if an infinite number of coupling constants is needed, are defined to be renormalizable in the modern sense. The difficulty, that is also discussed, is that the non trivial infinities are a priori only constrained to belong to the BRST cohomology in ghost number 0, which, because of the non validity of the generalized Kluberg-Stern and Zuber conjecture (taken as an example of a so called structural constraint), does not guarantee that they can be absorbed by redefinitions of coupling constants of an action extended by all possible off-shell observables. What good structural constraints might be in the general case and if they can be chosen in such a way as to guarantee renormalizability in the modern sense for all theories is left as an open question in [21, 22].

A clue to the answer to these questions can be found in [14, 15, 16]. Indeed, the fact that the divergences are such that they always provide a deformation of the solution of the master equation, implies in general that the non trivial first order deformations satisfy additional cohomological restrictions [23] besides belonging to the BRST cohomology. The problem with these restrictions is that they are non linear in the case of an arbitrary deformation. Recently [24], it has been shown that the lowest order additional restriction on the non trivial counterterms is in fact

[1] Note that it is not true that the antifield independent part of the cohomology of the differential $(S,\cdot)$ is off-shell gauge invariant, it is in general only weakly gauge invariant.

JHEP03(1999)010

linear: the counterterms must belong to the kernel of the antibracket map, which defines a linear subspace of the BRST cohomology.

As an (academical) example of how these higher order cohomological restrictions work, consider Yang-Mills theories with free[2] abelian gauge fields A^a_μ as in [21]. The BRST cohomology in ghost number zero contains the term [7]

$$K = f_{abc} \int d^n x \ F^{a\nu\mu} A^b_\mu A^c_\nu + 2A^{*a\mu} A^b_\mu C^c + C^{*a} C^b C^c \,,$$

for completely antisymmetric constants f_{abc}, so that this term is a potential counterterm. At the same time, the term $k^d \int d^n x \ C^*_d$ belongs to the BRST cohomology in ghost number -2. If we take the action $S_k = S + k^d \int d^n x \ C^*_d$, we have $1/2(S_k, S_k) = O(k^2)$. This implies according to the quantum action principle for the regularized theory that $1/2(\Gamma_k, \Gamma_k) = O(k^2)$ and then, at order 1 in $\hbar$ for the divergent part, that

$$(S_k, \Gamma^{(1)}_{k\ div}) = O(k^2)\,.$$

The k independent part of this equation gives the usual condition that the divergent part of the k independent effective action at first order must be BRST closed, $(S, \Gamma^{(1)}{}_{div}) = 0$, and contains in particular the candidate K above. The k linear part of this equation requires

$$\left(\left.\frac{\partial \Gamma^{(1)}_{k\ div}}{\partial k^d}\right|_{k=0}, S \right) + \left(\int d^n x \ C^*_d, \Gamma^{(1)}{}_{div} \right) = 0\,.$$

This condition eliminates the candidate K because

$$\left(\int d^n x \ C^*_d, K \right) = 2 f_{abd} \int d^n x \ A^{*a\mu} A^b_\mu + C^{*a} C^b$$

is not BRST exact but represents a non trivial BRST cohomology class in ghost number -1. Hence, there exists a purely cohomological reason why K cannot appear as a counterterm. Note that as soon as the abelian fields are coupled to matter fields, the functionals $\int d^n x \ C^*_d$ but also K do not belong to the BRST cohomology any more and the problem with this particular type of counterterms does not arise to begin with.

Another, non trivial example of how antifield dependent counterterms can be eliminated by higher order cohomological restrictions is discussed in the appendix. This example is physically relevant in the case of the standard model. In the main part of this paper however, we will focus on the general construction of a formalism to deal with higher order cohomological restrictions. By general, we mean that the construction is independent of the concrete gauge theory or type of instability

[2]By free, we mean that the abelian gauge fields have no couplings to matter fields, hence, they have no interactions at all. Their quantization is of course trivial and we know a priori that no counterterms are needed.

JHEP03(1999)010

under consideration and uses only arguments involving the (integrated, local and antifield dependent) BRST cohomology and the anticanonical structure of the antifield formalism.

A related problem, which is relevant in [16, 20, 21], is to provide a sensible definition of the Δ operator of the quantum Batalin-Vilkovisky master equation [9]. Indeed, its expression as a second order functional differential operator with respect to fields and antifields, obtained from formal path integral considerations, does not make sense when applied to local functionals. In [25, 26, 27, 28], the antifield formalism has been discussed in the context of explicit regularization and renormalization schemes and the related question of anomalies (assumed to be absent in [16, 20, 21]) has been adressed. In particular, well defined expressions for the regularized Δ operator are proposed at one loop level in [25] in the context of Pauli-Villars regularization and at higher orders in [28] for non-local regularization.

The purpose of the present paper is to answer the questions raised in [21, 22] and to show renormalizability in the modern sense for all gauge theories. We complete the general analysis of the absorption of divergences in the presence of possibly anomalous local or global symmetries independently of the form of the BRST cohomology of the theory. At the same time, a definition of the quantum Batalin-Vilkovisky Δ operator in the framework of dimensional renormalization is proposed.

This is done, on the classical level, by

- introducing higher order linear maps on the BRST cohomology, related to the Lie-Massey brackets [29] and constructed by a perturbative method using constant ghosts and their antifields, as in the construction [13] of the extended antifield formalism,

- identifying the constant ghosts, coupling all possible (local integrated) observables of the theory, with (generalized) essential coupling constants,

- showing that the theory to which all observables are coupled satisfies an extended master equation at the classical level with properties similar to those of the quantum master equation,

- proving that the solution to the extended master equation is complete in the sense that the cohomology associated to this solution can be obtained by derivation with respect to the coupling constants,

- showing how to extend the solution to which an arbitrary cocycle has been added into a complete deformed solution satisfying the same extended master equation and the relation of this deformation to field-antifield and coupling constant redefinitions.

JHEP03(1999)010

On the quantum level, in the context of dimensional regularization in possibly anomalous theories,[3] we show

- in the case of an invariant regularization that, order by order, the divergences are cocycles in ghost number 0 and as such, they can be absorbed by redefinitions of this solution determined by coupling constant and anticanonical field-antifield renormalizations in such a way that both the subtracted and the effective action satisfy the extended master equation,

- in the general case of a non invariant scheme, that the one loop divergences and anomalies are cocycles in ghost numbers 0 and 1,

- how the one loop divergences can be absorbed by field-antifield and coupling constant redefinitions, up to a finite BRST breaking counterterm, chosen in such a way that only non trivial anomalies appear,

- how to continue to higher orders by redefinitions preserving the extended master equation up to BRST breaking counterterms in such a way that the anomalous breaking of the extended Zinn-Justin equation is entirely determined by the cohomology of the extended BRST differential in ghost number 1,

- the consequences for the definition of the Batalin-Vilkovisky Δ operator in dimensional renormalization.

In order to relate the terminology used in this paper with the one coming from the study of the renormalization of non abelian Yang-Mills theories [2, 4] (see for instance [35, 36, 37] for reviews), we note that what is called extended master equation here corresponds to a generalized Slavnov-Taylor identity. By generalized, we mean first of all the definition of this identity in theories with arbitrary gauge structure as proposed by Batalin and Vilkovisky [9], then the extension to include the case of (a closed subset of) global symmetries [12, 13] and finally, the extension proposed here to include all the generalized observables of the theory. What is called (local) BRST cohomology corresponds to the cohomology of the generalized linearized Slavnov-Taylor operator S_L acting in the space of (integrated) polynomials in the fields, the sources and their derivatives.

The main, purely classical part of the paper adresses the question of how to define stability[4] in this generalized context and how to construct a stable classical

[3] As shown in [30, 31, 32, 33, 34] dimensional renormalization can deal consistently with anomalies if evanescent terms are taken into account properly. The same mechanism will be used here in the context of the Batalin-Vilkovisky formalism to show that the evanescent breaking terms of the regularized master equation are responsible for a non trivial Δ operator, even though "$\delta(0)'' = 0$.

[4] Stability is defined for instance in [37] as "the dimension of the cohomology space of the S_L operator in the Faddeev-Popov neutral sector should be equal to the number of physical parameters of the classical action".

JHEP03(1999)010

action *independently of power counting restrictions.* This action then serves as a starting point for the quantum theory. This is the translation, in the language of algebraic renormalization, of showing what is called *renormalizability in the modern sense* for generic gauge theories by the authors of [21].

The investigation relies heavily on the anticanonical structure of variational theories discovered by Zinn-Justin [4] in the context of non abelian Yang-Mills theories (see also [5]) and reintroduced by Batalin and Vilkovisky in the general context, in analogy with the generalized Poisson bracket of the Hamiltonian formalism [38].

The quantum part of the paper first shows that if the regularization scheme is invariant, the generalized Slavnov-Taylor identities hold at the quantum level. In the cases when dimensional regularization is not an invariant scheme for the symmetries under considerations, it is shown that BRST breaking counterterms can be choosen in such a way that the breakings of the generalized Slavnov-Taylor identity for the renormalized effective action is described to each order by the cohomology of the Slavnov-Taylor operator in ghost number 1. The investigation is then used to derive, in the context of dimensional renormalization at the level of the regularized classical action with all its counterterms, a definition of the Batalin-Vilkovisky Δ operator [9], which is an operator arising naturally in formal path integral manipulations (see also [39]).

2. Local BRST cohomology, higher order maps and deformations

2.1 The (extended) antifield formalism

The central object of the formalism is a solution S to the classical master equation

$$\frac{1}{2}(S,S)=0\,. \tag{2.1}$$

This solution is obtained by constructing a (possibly reducible) generating set of non trivial Noether identities of the equations of motion $\delta L_0/\delta\phi^i=0$ associated to the classical action $S_0=\int d^nxL_0$, as well as generating sets of non trivial reducibility identities for the Noether identities, of non trivial reducibility identities of the second stage for the previous reducibility identities... The original set of fields $\{\phi^i\}$ is extended to the set $\{\phi^A\}$ by introducing in addition

(i) ghost fields for the non trivial Noether identities,

(ii) ghosts for ghosts associated to the non trivial reducibility identities of Noether identities,

(iii) ghosts for ghosts for ghosts for the second stage non trivial reducibility identities...

JHEP03(1999)010

(iv) antifields ϕ_A^* associated to all of the above fields. In the following, we will denote the fields and antifields collectively by z.

In the classical theory, the relevant space is the space of local functionals $\mathcal{F}^*$ in the fields and antifields. Under appropriate vanishing conditions on the fields, antifields and their derivatives at infinity, this space is isomorphic to the space of functions in the fields, the antifields and a finite number of their derivatives, up to total divergences (see e.g. [19] and references therein for more details). It is an odd graded Lie algebra with respect to the antibracket $(\cdot,\cdot)$, the grading being given by the ghost number, and the antibracket being defined by considering the fields and antifields as canonically conjugate. Note however that the product of two local functionals is not a local functional, and that there is no direct definition of the second order Δ operator [9] in this space.

If one requires the solution S of the master equation (2.1) to be in ghost number 0, Grassmann even and proper, i.e. to contain in addition to the starting point action S_0 the gauge transformations related to the generating set of non trivial Noether identities as well as the various reducibility identities in a canonical way, one can show [9, 14, 17, 18, 10] existence and locality of this solution, with uniqueness holding up to canonical field-antifield redefinitions. The BRST differential is then $s = (S, \cdot)$, so that $(\mathcal{F}^*, (\cdot,\cdot), s)$ is a graded differential Lie algebra with an inner differential.

This construction can be extended to include a closed sub-algebra of non trivial global symmetries, by coupling their generators with constant ghosts and introducing constant antifields, if due care is taken of higher order conservation laws [13]. The formalism then allows to control through the master equation both local and a subset of global symmetries, or, in the absence of gauge symmetries, the subset of global symmetries alone. This extension will be a part of a further extension done in the next section. Indeed, the generators of the global symmetries correspond to local BRST cohomology classes in negative ghost numbers [19]. Here, we will couple the local BRST cohomology classes in *all* the ghost numbers.

A gauge fixed action with well defined propagators as a starting point for perturbation theory is obtained by introducing a non minimal sector, which has trivial cohomology [40], followed by an anticanonical field-antifield transformation. The new antifields are not set to zero, but kept as sources in the generating functionals of Green's functions in order to control the renormalization of the symmetries. In the considerations below, we will only be interested in aspects related to the local BRST cohomology. The local BRST cohomology of two formulations of the theory related by anticanonical field-antifield transformations are isomorphic. This means in particular that all the results to be derived are independent from the choice of the gauge fixation. This is the reason why we will not explicitly make the steps corresponding to the gauge fixation, i.e. introduction of the non minimal sector and transformation to the gauge fixed basis, although they are understood in the manipulations involving the Green's functions.

JHEP03(1999)010

2.2 Higher order maps in BRST cohomology from homological perturbation theory

In this section, we construct a generating functional $\tilde{\mathcal{S}}$ for higher order maps in BRST cohomology using homological perturbation theory [41] (see [10] for details on homological perturbation theory in the context of the Batalin-Vilkovisky formalism) by adapting the construction of the extended antifield formalism [13].

For any functional $A \in \mathcal{F}^*$, the equation $(S, A) = 0$ implies $A = S_A\lambda^A + (S, B)$, where λ^A is independent of the fields and anti-fields, but can depend on the coupling constants of the theory, with $S_A\lambda^A + (S, B) = 0$ iff $\lambda^A = 0$. In other words, we suppose that $[S_A]$ is a basis of $H^*(s, \mathcal{F})$ over the ring of functions in the coupling constants.

For each S_A of the above basis, we introduce a constant "ghost" ξ^A and a constant "antifield" ξ^*_A such that $gh\ \xi^A = -gh\ S_A$, $gh\ \xi^*_A = -gh\ \xi^A - 1$. We consider the space $\mathcal{E}$ of functionals $\mathcal{A}$ of the form

$$\mathcal{A} = A[\phi, \phi^*, \xi] + \xi^*_A\lambda^A(\xi)\,, \tag{2.2}$$

i.e. $\mathcal{A}$ contains a local functional A which admits in addition to the dependence on the coupling constants, a dependence on the constant ghosts ξ^A, and a non integrated piece linear in the constant antifields ξ^*_A depending only on the constant ghosts (and the coupling constants).

The differential $\tilde{\delta}$ is defined by $\tilde{\delta} A = (S, A)$, $\tilde{\delta}\xi^*_A = S_A$, and $\tilde{\delta}\xi^A = 0$.

Lemma 1 *The cohomology of $\tilde{\delta}$ is trivial, $H(\tilde{\delta}, \mathcal{E}) = 0$.*

Proof. Indeed, $\tilde{\delta}\mathcal{A} = 0$ gives $(S, A) + S_A\lambda^A = 0$, and hence $\lambda^A = 0$, so that $\mathcal{A} = A = S_A\mu^A + (S, B) = \tilde{\delta}(B + \xi^*_A\mu^A)$. □

We define the resolution degree to be the degree in the ghosts ξ^A, which implies that $\tilde{\delta}$ is of degree 0.

The extended antibracket is defined by

$$\begin{aligned}(\cdot,\cdot\tilde{)} &= (\cdot,\cdot) + (\cdot,\cdot)_\xi \\ &= (\cdot,\cdot) + \frac{\partial^R}{\partial\xi^A}\frac{\partial^L}{\partial\xi^*_A} - \frac{\partial^R}{\partial\xi^*_A}\frac{\partial^L}{\partial\xi^A}\end{aligned} \tag{2.3}$$

and satisfies the same graded antisymmetry and graded Jacobi identity as the usual antibracket. The extended antibracket has two pieces, the old piece $(\cdot,\cdot)$, which is of degree 0, and the new piece $(\cdot,\cdot)_\xi$, which is of degree -1.

Theorem 1 *There exists a solution $\tilde{\mathcal{S}} \in \mathcal{E}$ of ghost number 0 to the master equation*

$$\frac{1}{2}(\tilde{\mathcal{S}}, \tilde{\mathcal{S}}\tilde{)} = 0\,. \tag{2.4}$$

with initial condition $\tilde{\mathcal{S}} = S + S_A\xi^A + \ldots$, where the dots denote terms of resolution degree higher or equal to 2. The cohomology of the differential $\tilde{s} = (\tilde{\mathcal{S}}, \cdot\tilde{)}$ in $\mathcal{E}$ is trivial.

JHEP03(1999)010

Proof. The proofs are by now standard and follow the lines of [10]. Let $\tilde{\mathcal{S}}^1 = S + S_A\xi^A$. Note that $\tilde\delta$ is the piece of degree 0 in $(\tilde{\mathcal{S}}^1,\cdot\tilde{)}$ and that $(\mathcal{S}_k,\cdot\tilde{)}$ has no piece in degree 0 if $k\geq 2$. Suppose that we have constructed $\tilde{\mathcal{S}}^k = S+\mathcal{S}_1+\cdots+\mathcal{S}_k$ up to degree $k\geq 1$, with

$$\frac{1}{2}(\tilde{\mathcal{S}}^k,\tilde{\mathcal{S}}^k\tilde{)} = \mathcal{R}_{k+1} + O(k+2)\,. \tag{2.5}$$

The identity $0 = (\tilde{\mathcal{S}}^k, \frac{1}{2}(\tilde{\mathcal{S}}^k,\tilde{\mathcal{S}}^k\tilde{)}\tilde{)}$ then implies, at order $k+1$, that $\tilde\delta\mathcal{R}_{k+1}=0$, so that there exists $\mathcal{S}_{k+1}$ such that $\mathcal{R}_{k+1}+\tilde\delta\mathcal{S}_{k+1}=0$. The action $\tilde{\mathcal{S}}^{k+1}=\tilde{\mathcal{S}}^k+\mathcal{S}_{k+1}$ then satisfies

$$\frac{1}{2}(\tilde{\mathcal{S}}^{k+1},\tilde{\mathcal{S}}^{k+1}\tilde{)} = \mathcal{R}_{k+1}+\tilde\delta\mathcal{S}_{k+1}+O(k+2) = O(k+2)\,, \tag{2.6}$$

so that the construction can be continued recursively.

For the second part of the theorem, we develop a cocycle $\mathcal{A}$ according to the resolution degree, $\mathcal{A}=\mathcal{A}_M+\mathcal{A}_{M+1}+\cdots$, with $M\geq 0$. At lowest order the condition $(\tilde{\mathcal{S}},\mathcal{A}\tilde{)}=0$ implies $\tilde\delta\mathcal{A}_M=0$ which gives $\mathcal{A}_M=\tilde\delta\mathcal{B}_M$ for some $\mathcal{B}_M$. The cocycle $\mathcal{A}-(\tilde{\mathcal{S}},\mathcal{B}_M\tilde{)}$ is equivalent to $\mathcal{A}$, but starts at order $M+1$. Going on in the same way, one can absorb all the terms so that $\mathcal{A}=(\tilde{\mathcal{S}},\mathcal{B}\tilde{)}$ for some $\mathcal{B}\in\mathcal{E}$. □

The solution $\tilde{\mathcal{S}}$ is of the form

$$\tilde{\mathcal{S}} = S + \sum_{k=1} S_{A_1\ldots A_k}\xi^{A_1}\cdots\xi^{A_k} + \sum_{m=2}\xi^*_B f^B_{A_1\ldots A_m}\xi^{A_1}\cdots\xi^{A_m}\,, \tag{2.7}$$

which implies the graded symmetry of the generalized structure constants $f^B_{A_1\ldots A_m}$ and the functionals $S_{A_1\ldots A_k}$. The ξ^*_A independent part of the master equation (2.4) gives, at resolution degree $r\geq 1$, the relations

$$\begin{aligned}(S,S_{A_1\ldots A_r}) &+ \sum_{k=1}^{r-1}\frac{1}{2}(S_{(A_1\ldots A_k},S_{A_{k+1}\ldots A_r)})(-)^{(A_1+\ldots+A_k)(A_{k+1}\ldots A_r+1)} + \\ &+ \sum_{k=1}^{r-1} kS_{(A_1\ldots A_{k-1}|B|}f^B_{A_k\ldots A_r)} = 0\,,\end{aligned} \tag{2.8}$$

where ( ) denotes graded symmetrization. The first relations read explicitly

$$(S,S_{A_1}) = 0\,, \tag{2.9}$$

$$(S,S_{A_1A_2}) + \frac{1}{2}(S_{A_1},S_{A_2})(-)^{A_1(A_2+1)} + S_Bf^B_{A_1A_2} = 0\,, \tag{2.10}$$

$$(S,S_{A_1A_2A_3})+(S_{(A_1},S_{A_2A_3)})(-)^{A_1(A_2+A_3+1)}+S_Bf^B_{A_1A_2A_3}+2S_{(A_1|B|}f^B_{A_2A_3)} = 0\,, \tag{2.11}$$

$$\vdots$$

The ξ^*_A dependent part of the master equation (2.4) gives, for $r\geq 3$, the generalized Jacobi identities

$$\sum_{m=2}^{r-1} mf^C_{(A_1\ldots A_{m-1}|B|}f^B_{A_m\ldots A_r)} = 0\,, \tag{2.12}$$

JHEP03(1999)010

the first identities being

$$2f^C_{(A_1|B|}f^B_{A_2A_3)} = 0\,, \tag{2.13}$$

$$2f^C_{(A_1|B|}f^B_{A_2A_3A_4)} + 3f^C_{(A_1A_2|B|}f^B_{A_3A_4)} = 0\,, \tag{2.14}$$

$$\vdots$$

The above solution $\tilde{\mathcal{S}}$ is not unique. For a given initial condition, there is at each stage of the construction of $\tilde{S}$, for $k \geq 2$, the liberty to add the exact term $\tilde{\delta}K_k$ to $\tilde{\mathcal{S}}_k$. While this liberty will not affect the structure constants of order k, since a $\tilde{\delta}$ exact term does not involve a ξ^* dependent term, it will in general affect the structure constants of order strictly higher than k.

Furthermore, there is a freedom in the choice of the initial condition: instead of $S_1 = S_A\xi^A$, one could have chosen $S_1' = \sigma_A^B S_B \xi^A + (S, K_A)\xi^A$ with an invertible matrix σ_A^B. If we consider the following anticanonical redefinitions:

$$z' = \exp(\cdot, K_A\xi^A)z, \tag{2.15}$$

$$\xi'^B = \sigma_A^B \xi^A\,, \qquad \xi'^*_B = \sigma^{-1}{}^A{}_B \xi^*_A\,, \tag{2.16}$$

we have that $S + S_1' = S(z') + S_B(z')\xi'^B + O(\xi^2)$. We can then consider the solution $\tilde{\mathcal{S}}'$ in terms of the new variables. This is equivalent to taking as initial condition $S(z') + S_B(z')\xi'^B$ and making the same choices for the terms of degree higher than 2 in the new variables than we did before in the old variables. It is thus always possible to make the choices in the construction of $\tilde{\mathcal{S}}$ for $k \geq 2$ in such a way that the structure constants $f^B_{A_1\ldots A_m}$ do not depend on the choice of representatives for the cohomology classes and transform tensorially with respect to a change of basis in $H^*(s, \mathcal{F})$.

Hence, we have shown

Theorem 2 *Associated to a solution $\tilde{\mathcal{S}}$ of the master equation (2.4), there exist multi-linear, graded symmetric maps in cohomology, defined through the structure constants $f^B_{A_1\ldots A_r}$:*

$$l_r : \wedge^r H^*(s) \longrightarrow H^*(s) \tag{2.17}$$

$$l_r\,([S_{A_1}], \ldots, [S_{A_r}]) = -r[S_B] f^B_{A_1\ldots A_r}\,. \tag{2.18}$$

Remark. For a given $r \geq 2$, let us suppose that the $[S_{A_1}], \ldots, [S_{A_r}]$ are such that the structure constants with strictly less than r indices vanish, for all choices of A_i's. From the identity (2.8), it then follows that

$$\sum_{k=1}^{r-1} \frac{1}{2}(S_{(A_1\ldots A_k}, S_{A_{k+1}\ldots A_r)})(-)^{(A_1+\ldots+A_k)(A_{k+1}\ldots A_r+1)} = -rS_B f^B_{A_1\ldots A_r} - (S, S_{A_1\ldots A_r})\,. \tag{2.19}$$

JHEP03(1999)010

We thus see that, under the above assumption,

$$l_r([S_{A_1}],\dots,[S_{A_r}]) = \left[\sum_{k=1}^{r-1}\frac{1}{2}(S_{(A_1\dots A_k},S_{A_{k+1}\dots A_r)})(-)^{(A_1+\dots+A_k)(A_{k+1}\dots A_r+1)}\right]. \qquad (2.20)$$

By comparing with the invariant definitions in [29], we identify the maps l_r, under the above assumption, as the value of the r-place Lie-Massey product $[[S_{A_1}],\dots,[S_{A_r}]]$ for the defining system $\{S_{A_{i_1}\dots A_{i_k}},\ k=1,\dots,r-1,\ 1\leq i_1<\dots<i_k\leq r\}$. $\square$

2.3 Coupling constants

Differentiating the master equation (2.1) with respect to a coupling constant g, implies that $\left(S,\frac{\partial^R S}{\partial g}\right)=0$, so that $\left[\frac{\partial^R S}{\partial g}\right]\in H^0(s)$.

Let us adapt the considerations in [42] to the present context.

Definition 1 *A set of coupling constants g^i is essential iff the relation $\frac{\partial^R S}{\partial g^i}\lambda^i = (S,\Xi)$ implies $\lambda^i=0$, where λ^i may depend on all the couplings of the theory.*

It follows that for essential couplings the $\left[\frac{\partial^R S}{\partial g^i}\right]$ are linearly independent in $H^0(s)$ and that essential couplings stay essential after anticanonical field-antifield redefinitions.

In the following, we suppose that S depends only on essential couplings. Note that this can always be achieved. If among the couplings, there is g such that $\frac{\partial^R S}{\partial g} = \frac{\partial^R S}{\partial g^{\bar i}}\lambda^{\bar i} + (S,\Xi)$, one can show that the dependence of S on g can be absorbed by an anticanonical, g dependent field-antifield redefinition and a g dependent redefinition of the other coupling constants $g^{\bar i}$.

Since the $\left[\frac{\partial^R S}{\partial g^i}\right]$ are linearly independent, one can construct a basis $[S_A]$ of $H^*(s)$ such that the $\left[\frac{\partial^R S}{\partial g^i}\right]$ are the first elements. Let us denote the remaining elements by $[S_\alpha]$, $\{[S_A]\}=\left\{\left[\frac{\partial^R S}{\partial g^i}\right],[S_\alpha]\right\}$. The construction of the generating functional $\tilde{\mathcal{S}}$ then starts with $S(g^i)+\frac{\partial^R S}{\partial g^i}\xi^i + S_\alpha\xi^\alpha$.

Consider the action $\bar S = S(g^i+\xi^i)$. A basis of the cohomology of $\bar S$ is given by $\left\{\left[\frac{\partial^R \bar S}{\partial g^i}\right],[\bar S_\alpha]\right\}$, with associated differential $\bar{\tilde\delta} = (\bar S,\cdot)+\frac{\partial^R\bar S}{\partial\xi^i}\frac{\partial^L}{\partial\xi^*_i}+\bar S_\alpha\frac{\partial^L}{\partial\xi^*_\alpha}$, which is acyclic in the space where the only dependence on ξ^i is through the combination $g^i+\xi^i$. If we take as starting point the action $\bar S+\bar S_\alpha\xi^\alpha$ and start the perturbative construction of the solution of the master equation, with resolution degree the degree in the ghost ξ^α alone, the ghosts ξ^i only appear through the combination $g^i+\xi^i$, because of the properties of $\bar{\tilde\delta}$. The solution $\bar{\tilde{\mathcal{S}}}$ will then be of the form

$$\bar{\tilde{\mathcal{S}}} = \bar S + \sum_{k=1}\bar S_{\alpha_1\dots\alpha_k}\xi^{\alpha_1}\cdots\xi^{\alpha_k} + \sum_{m=2}(\xi^*_\beta \bar f^\beta_{\alpha_1\dots\alpha_m} + \xi^*_i \bar f^i_{\alpha_1\dots\alpha_m})\xi^{\alpha_1}\cdots\xi^{\alpha_m}, \qquad (2.21)$$

where the $\bar S_{\alpha_1\dots\alpha_k}$, $\bar f^i_{\alpha_1\dots\alpha_m}$, $\bar f^i_{\alpha_1\dots\alpha_m}$ depend on the combination $g^i+\xi^i$.

Now, the solution $\bar{\tilde{\mathcal{S}}}$ satisfies the initial condition $\bar{\tilde{\mathcal{S}}}^1 = S(g)+\frac{\partial^R S}{\partial g^i}\xi^i + S_\alpha\xi^\alpha$ in the old resolution degree and the master equation (2.4). We can then derive the

JHEP03(1999)010

higher order maps l_r from the solution (2.21) and get

$$l_r\left(\left[\frac{\partial^R S}{\partial g^{i_1}}\right],\ldots,\left[\frac{\partial^R S}{\partial g^{i_n}}\right],[S_{\alpha_{n+1}}],\ldots,[S_{\alpha_r}]\right)=$$

$$=-r[S_\beta]\frac{\partial^{R^n}\bar{f}^{\beta}_{\alpha_{n+1}\ldots\alpha_r}}{\partial g^{i_n}\ldots\partial g^{i_1}}(g)-r\left[\frac{\partial^R S}{\partial g^j}\right]\frac{\partial^{R^n}\bar{f}^{j}_{\alpha_{n+1}\ldots\alpha_r}}{\partial g^{i_n}\ldots\partial g^{i_1}}(g)\,. \quad (2.22)$$

In the following, we will make the redefinition $g^i+\xi^i\longrightarrow\xi^i$, and identify the essential couplings with some of the constant ghosts. One could of course have done the converse, i.e. identify some of the constant ghosts with the essential couplings, but since the anticanonical structure between the constant ghosts and their constant antifields turns out to be crucial, we prefer to do the former. The remaining constant ghosts can then be considered as generalized essential coupling constants since they couple the remaining BRST cohomology classes, which play the role of generalized observables in this formalism.

2.4 Decomposition of $\tilde{s}$

The space $\mathcal{E}$ admits the direct sum decomposition $\mathcal{E}=F\oplus G$, where $F=\mathcal{E}|_{\xi^*=0}$ is the space of functionals in the field and antifields with ξ dependence, but no ξ^* dependence, while G is the space of power series in ξ with a linear ξ^* dependence.

The differential $\tilde{s}$ in $\mathcal{E}$ induces two well-defined differentials, $\bar{s}$ in F and s_Q in G given explicitly by

$$\bar{s}=(S(\xi),\cdot)-(-)^{D+1}f^D\frac{\partial^L}{\partial\xi^D} \quad (2.23)$$

and

$$s_Q=(Q,\cdot)_\xi,\ \ Q=\xi^*_C f^C(\xi)\,. \quad (2.24)$$

Indeed, for $\mathcal{A}=A(\xi)+\xi^*_D\lambda^D(\xi)$, the master equation (2.4) implies $(\tilde{\mathcal{S}},(\tilde{\mathcal{S}},\mathcal{A}\tilde{)}\tilde{)}=0$ and hence $(\tilde{\mathcal{S}},\bar{s}A(\xi)+s_Q\xi^*_D\lambda^D(\xi)\tilde{)}=0$ and then $(\bar{s})^2A(\xi)+(s_Q)^2\xi^*_D\lambda^D(\xi)=0$, which splits into two equations because the decomposition of $\mathcal{E}$ is direct.

Theorem 3 *The cohomology groups $H(\bar{s},F)$ and $H(s_Q,G)$ are isomorphic.*

Proof. Let us take $\tilde{S}=S(\xi)+\xi^*_C f^C(\xi)$, $\mathcal{A}=A(\xi)+\xi^*_C\lambda^C(\xi)$ and $\mathcal{B}=B(\xi)+\xi^*_C\mu^C(\xi)$. The extended master equation (2.4) can be written compactly as

$$\frac{1}{2}(S(\xi),S(\xi))+\frac{\partial^R S(\xi)}{\partial\xi^C}f^C=0\,, \quad (2.25)$$

$$\frac{1}{2}(\xi^*_C f^C(\xi),\xi^*_D f^D(\xi))_\xi=0\,, \quad (2.26)$$

so that (2.25) summarizes (2.8) and (2.26), which is equal to $\frac{1}{2}(Q,Q)_\xi=0$, or explicitly $\frac{\partial^R f^D(\xi)}{\partial\xi^C}f^C(\xi)=0$, summarizes the generalized Jacobi identities (2.12).

JHEP03(1999)010

The triviality of the cohomology of $(\tilde{S}, \cdot\tilde{)}$ in the space $\mathcal{E}$, i.e. the fact that the general solution to $(\tilde{S}, \mathcal{A}\tilde{)} = 0$ is $\mathcal{A} = (\tilde{S}, \mathcal{B}\tilde{)}$, is expressed explicitly through the fact that the general solution to the set of equations

$$\begin{cases} (S(\xi), A(\xi)) + \dfrac{\partial^R S(\xi)}{\partial \xi^C} \lambda^C(\xi) - (-)^{D+1} f^D \dfrac{\partial^L A(\xi)}{\partial \xi^D} = 0\,, \\ (\xi^*_C f^C(\xi), \xi^*_D \lambda^D(\xi))_\xi = 0\,, \end{cases} \tag{2.27}$$

is given by

$$\begin{cases} A(\xi) = (S(\xi), B(\xi)) + \dfrac{\partial^R S(\xi)}{\partial \xi^C} \mu^C(\xi) - (-)^{D+1} f^D \dfrac{\partial^L B(\xi)}{\partial \xi^D}\,, \\ \xi^*_E \lambda^E(\xi) = (\xi^*_C f^C(\xi), \xi^*_D \mu^D(\xi))_\xi\,. \end{cases} \tag{2.28}$$

The cocycle condition $\bar{s}A(\xi) = 0$ is given by

$$(S(\xi), A(\xi)) - (-)^{D+1} f^D \frac{\partial^L A(\xi)}{\partial \xi^D} = 0\,, \tag{2.29}$$

while the coboundary condition $A(\xi) = \bar{s}B(\xi)$ is given by

$$A(\xi) = (S(\xi), B(\xi)) - (-)^{D+1} f^D \frac{\partial^L B(\xi)}{\partial \xi^D}\,. \tag{2.30}$$

But according to (2.27) and (2.28), where we put $\lambda^D(\xi) = 0$, we have

$$\bar{s}A(\xi) = 0 \Longleftrightarrow A(\xi) = \bar{s}B(\xi) + \frac{\partial^R S(\xi)}{\partial \xi^C} \mu^C(\xi)\,, \tag{2.31}$$

under the condition

$$(\xi^*_C f^C(\xi), \xi^*_D \mu^D(\xi))_\xi = 0 \tag{2.32}$$

on $\mu^D(\xi)$. The constraint (2.32) requires $\xi^*_D \mu^D(\xi)$ to be a s_Q cocycle.

In order to compute the cohomology $H(\bar{s}, F)$, we have to analyze when the decomposition in (2.31) is direct, i.e. we have to analyze when $\frac{\partial^R S(\xi)}{\partial \xi^C} \mu^C(\xi)$ is a $\bar{s}$ coboundary. We thus need to solve the equation $\frac{\partial^R S(\xi)}{\partial \xi^C} \mu^C(\xi) = -\bar{s}\bar{B}$ under the condition $s_Q \xi^*_D \mu^D(\xi) = 0$. But this corresponds precisely to equation (2.27), with A replaced by $\bar{B}$ and $\lambda(\xi)$ by $\mu(\xi)$, whose general solution according to (2.28) is given by $\bar{B} = \bar{s}C(\xi) + \frac{\partial^R S(\xi)}{\partial \xi^C} \nu^C(\xi)$ and $\xi^*_D \mu^D(\xi) = s_Q \xi^*_E \nu^E(\xi)$. (In other words, we are using again the triviality of $H(\tilde{s}, \mathcal{E})$.)

The map $m(\xi^*_D \mu^D(\xi)) = \frac{\partial^R S(\xi)}{\partial \xi^C} \mu^C(\xi)$ maps s_Q cocyles to $\bar{s}$ cocyles. We have just proved above that it maps s_Q coboundaries to $\bar{s}$ coboundaries, so that the map induced by m in cohomology,

$$m : H(s_Q, G) \longrightarrow H(\bar{s}, F)\,, \tag{2.33}$$

$$m([\xi^*_D \mu^D(\xi)]) = \left[\frac{\partial^R S(\xi)}{\partial \xi^C} \mu^C(\xi)\right] \tag{2.34}$$

is well-defined and injective. Surjectivity of the induced map follows from (2.31). $\square$

JHEP03(1999)010

Discussion.

(i) In order to compare the starting point cohomology $H^*(s, \mathcal{F})$ with the cohomology $H^*(\bar{s}, F)$, we can put the additional couplings ξ^α to zero in (2.31). The cocycle condition then reduces to the standard cocycle condition of the non extended formalism, $sA_{\xi^\alpha=0} = 0$. The same operation in the general solution gives $A_{\xi^\alpha=0} = sB_{\xi^\alpha=0} + \frac{\partial^R S}{\partial \xi^i}\mu^i_{\xi^\alpha=0} + S_\alpha \mu^\alpha_{\xi^\alpha=0}$. Contrary to the ordinary s cohomology, the coefficients $\mu^A_{\xi^\alpha=0}$ are not free however, but they come from μ^A's which are constrained to satisfy the cocyle condition (2.32). In particular, at order 1 in the new couplings ξ^α, (2.32) implies that $\mu^\alpha_{\xi^\alpha=0}$ is in the kernel of the map l_2, $f^A_{\beta\alpha}\mu^\alpha_{\xi^\alpha=0} = 0$, which is precisely the condition used in the examples of the introduction to eliminate elements of the cohomology of the theory without the additional couplings. We thus see that the cohomology has become smaller through the introduction of the additional couplings because the extended differential encodes higher order cohomological restrictions.

(ii) At first sight, it might seem a little strange to introduce new couplings in order to get information on the renormalization of the theory without these couplings: that it is convenient and extremely useful to do so was already realized in the original papers [2] on the subject: the additional (space-time dependent) couplings in these papers are just the sources of the BRS transformations, and can of course be set to zero after renormalization, if one is only interested in the renormalization of the effective action itself.

(iii) The result (2.31) implies also that the $\bar{s}$ cohomology is contained completely in the solution $S(\xi)$ and can be obtained from it by applying $\frac{\partial^R \cdot}{\partial \xi^A}\lambda^A(\xi)$, where the coefficients $\lambda^A(\xi)$ are constrained to satisfy (2.32). In this framework, this is what replaces the concept of stability as discussed in the introduction.

2.5 "Quantum" Batalin-Vilkovisky formalism on the classical level and deformations

If we define

$$\Delta_c = (-)^D f^D \frac{\partial^L}{\partial \xi^D}\,, \tag{2.35}$$

on F, the following properties of the quantum Batalin-Vilkovisky formalism hold in F: the operator Δ_c is nilpotent,

$$\Delta_c{}^2 = 0\,, \tag{2.36}$$

(as a consequence of (2.26) or (2.12)). Furthermore,

$$\Delta_c(A(\xi), B(\xi)) = \Big(\Delta_c A(\xi), B(\xi)) + (-)^{|A|+1}(A(\xi), \Delta_c B(\xi)\Big)\,. \tag{2.37}$$

JHEP03(1999)010

To the standard solution of the master equation S in $\mathcal{F}$ corresponds in F the solution $S(\xi)$ of the extended master equation

$$\frac{1}{2}(S(\xi), S(\xi)) + \Delta_c S(\xi) = 0\,, \tag{2.38}$$

(which is just rewriting (2.25) using the definition of Δ_c). Since

$$\bar{s} = (S(\xi), \cdot) + \Delta_c\,, \tag{2.39}$$

then the $\bar{s}$ cohomology corresponds to the quantum BRST cohomology σ discussed in [40, 10]. Theorem 3 shows how to compute the quantum BRST cohomology out of the standard BRST cohomology and the higher order maps encoded in s_Q.

In this analogy, putting $\xi = 0$ corresponds to the classical limit $\hbar \longrightarrow 0$ of the quantum Batalin-Vilkovisky formalism.

Note however that

(i) the space F is not directly an algebra, because the product of two local functionals is not well defined, contrary to the formal discussion of the quantum Batalin-Vilkovisky formalism, where one assumes the space to be an algebra,

(ii) the above "quantum" Batalin-Vilkovisky formalism is purely classical and depends only on the BRST cohomology and the higher order maps of the theory.[5]

We consider now one parameter deformations of the extended master equation (2.38), i.e. in the space $F[t]$ of power series in t with coefficients that belong to F, we want to construct $S_t(\xi) = S(\xi) + tS_1(\xi) + t^2 S_2(\xi) + \cdots$ such that

$$\frac{1}{2}(S_t(\xi), S_t(\xi)) + \Delta_c S_t(\xi) = 0\,. \tag{2.40}$$

A deformation $S_t(\xi) = S(\xi) + tS_1(\xi)$ to first order in t, i.e. such that $\frac{1}{2}(S_t(\xi), S_t(\xi)) + \Delta_c S_t(\xi) = O(t^2)$, is called an infinitesimal deformation. The term linear in t of an infinitesimal deformation, $S_1(\xi)$, is a cocycle of the extended BRST differential $\bar{s}$. If $S_1(\xi)$ is a $\bar{s}$ coboundary, we call the infinitesimal deformation trivial, while the parts of $S_1(\xi)$ corresponding to the $\bar{s}$ cohomology are non trivial.

Theorem 4 *Every infinitesimal deformation of the solution S to the extended master equation can be extended to a complete deformation S_t. This extension is obtained by performing a t dependent anticanonical field-antifield redefinition $z \to z'$, by performing a t dependent coupling constant redefinition $\xi \to \xi'$, which does not affect Δ_c, and by adding to $S(z', \xi')$ a suitable extension determined by both coupling constant and the field-antifield redefinition and vanishing whenever the latter does.*

[5] Further details on this cohomological approach to the quantum Batalin-Vilkovisky formalism will be considered elsewhere.

JHEP03(1999)010

Furthermore, the deformed solution considered as a function of the new variables $S_t(z(z',\xi'),\xi(\xi'))$ satisfies the extended master equation in terms of the new variables and the cohomology $H(\bar{s}',F')$ of the differential $\bar{s}' = (S_t,\cdot)_{z'} + \Delta'_c$ in the space F' of functionals depending on z',ξ' is isomorphic to the cohomology $H(\bar{s},F)$.

Proof. Equation (2.31) and (2.32) imply that $S_1(\xi) = \bar{s}B + \frac{\partial^R S(\xi)}{\partial \xi^C}\mu^C(\xi)$ with $(\xi^*_D f^D(\xi), \xi^*_C\mu^C(\xi))_\xi = 0$. In other words, $S_1(\xi) = (\tilde{S}, B(\xi) + \xi^*_C\mu^C(\xi))\tilde{}$. In the extended space $\mathcal{E}$, with $z^\alpha = (\phi^B, \phi^*_B)$, consider the anticanonical transformation

$$\begin{aligned} z'^\alpha &= \exp t(\cdot, B(\xi) + \xi^*_C\mu^C(\xi))\tilde{}\; z^\alpha \\ &= \exp t(\cdot, B(\xi)) z^\alpha \\ &= z^\alpha + t(z^\alpha, B(\xi)) + O(t^2)\,, \end{aligned} \tag{2.41}$$

$$\begin{aligned} \xi'^A &= \exp t(\cdot, B(\xi) + \xi^*_C\mu^C(\xi))\tilde{}\; \xi^A \\ &= \exp t(\cdot, \xi^*_C\mu^C(\xi))_\xi\; \xi^A \\ &= \xi^A + t\mu^A(\xi) + O(t^2)\,, \end{aligned} \tag{2.42}$$

$$\begin{aligned} {\xi'_A}^* &= \exp t(\cdot, B(\xi) + \xi^*_C\mu^C(\xi))\tilde{}\; \xi^*_A \\ &= \xi^*_A - t\frac{\partial^L}{\partial \xi^A}(B(\xi) + \xi^*_C\mu^C(\xi)) + O(t^2)\,. \end{aligned} \tag{2.43}$$

Note that $z' = z'(z,\xi)$, $\xi' = \xi'(\xi)$ and ${\xi^*_A}' = {\xi^*_A}'(z,\xi,\xi^*) = g_A(z,\xi) + \xi^*_B g^B_A(\xi)$, for a function $g_A(z,\xi) = -t\frac{\partial^L B}{\partial \xi^A} + O(t^2)$ determined by (2.43) through both B and μ and a function $g^B_A(\xi) = \delta^B_A - t(-)^{A(B+1)}\frac{\partial^L \mu^B}{\partial \xi^A} + O(t^2)$ determined by (2.43) through μ alone.

The master equation (2.4) holds in any variables, and thus also in terms of the primed variables. If we denote functions in terms of the new variables by a prime, we get $\frac{1}{2}(\tilde{S}', \tilde{S}')\tilde{}_{z',\xi'} = 0$. Because the transformation is anticanonical, we also have

$$\frac{1}{2}(\tilde{S}', \tilde{S}')\tilde{}_{z,\xi} = 0\,. \tag{2.44}$$

Since

$$\tilde{S}' = S' + g_A f'^A + \xi^*_B g^B_A f'^A\,, \tag{2.45}$$

equation (2.44) splits into

$$\frac{1}{2}(S' + g_A f'^A, S' + g_A f'^A)_z + \frac{\partial^R}{\partial \xi^D}(S' + g_A f'^A) g^D_E f'^E = 0, \tag{2.46}$$

$$\frac{1}{2}(\xi^*_B g^B_A f'^A, \xi^*_D g^D_C f'^C)_\xi = 0\,. \tag{2.47}$$

We have

$$\left.\frac{d(S' + g_A f'^A)}{dt}\right|_{t=0} = (S(\xi), B(\xi)) + \Delta_c B + \frac{\partial^R S(\xi)}{\partial \xi^D}\mu^D = S_1(\xi)\,, \tag{2.48}$$

JHEP03(1999)010

and, because $\xi^*_E\mu^E(\xi)$ is a s_Q cocycle, the relation

$$g^D_C f'^C = f^D\,. \tag{2.49}$$

Indeed, if we consider the above canonical transformation with $B = 0$, that is $\exp t(\cdot, \xi^*\mu)_\xi$ alone, $\xi^*_C g^C_D f^D(\xi') = \xi^{*\prime}{}_D f^D(\xi') = \exp t(\cdot, \xi^*\mu)_\xi \xi^*_E f^E = \xi^*_E f^E$, because $(\xi^*_E f^E, \xi^*_G\mu^G)_\xi = 0$. This shows the first part of the theorem, with $S_t = S' + g_A f'^A$.

In order to prove the second part, we first note that

$$\Delta_c{}' = (-)^D f^D(\xi')\frac{\partial^L}{\partial \xi'^D} = (-)^D f^D(\xi')\frac{\partial^L \xi^C}{\partial \xi'^D}\frac{\partial^L}{\partial \xi^C} = \Delta_c\,, \tag{2.50}$$

because

$$g^C_D = \frac{\partial^L \xi^C}{\partial \xi'^D}\,. \tag{2.51}$$

Indeed we have $\delta^A_B = (\xi'^A, \xi^*_B{}')\tilde{}_{z',\xi'} = (\xi'^A, \xi^*_B{}')_\xi = \frac{\partial^L \xi'^A}{\partial \xi^C} g^C_B$. Together with (2.46), this implies

$$\frac{1}{2}(S' + g_A f'^A, S' + g_A f'^A)_{z'} + \Delta'_c(S' + g_A f'^A) = 0\,. \tag{2.52}$$

We then start from the relations

$$(\tilde{S}', \mathcal{A}')\tilde{}_{z',\xi'} = 0 \Longleftrightarrow \mathcal{A}' = (\tilde{S}', \mathcal{B}')\tilde{}_{z',\xi'}\,, \tag{2.53}$$

where $\mathcal{A}' = A' + \xi^{*\prime}{}_A\lambda'^A$ and $\mathcal{B}' = B' + \xi^{*\prime}{}_A\rho'^A$. These relations hold with the bracket taken in the old variables, because the transformation is anticanonical. Writing the resulting relations explicitly, using (2.49), we get that the set of relations

$$\begin{cases} (S' + g_A f'^A, A' + g_B\lambda'^B) + \dfrac{\partial^R}{\partial \xi^C}(S' + g_A f'^A) g^C_D \lambda'^D\,, \\ +(-)^A f^A \dfrac{\partial^L}{\partial \xi^A}(A' + g_B\lambda'^B) = 0\,, \\ \xi^*_A \dfrac{\partial^R g^A_B f'^B}{\partial \xi^C} g^C_D\lambda'^D + (-)^D g^D_C f'^C \dfrac{\partial^L}{\partial \xi^D}(\xi^*_A g^A_B \lambda'^B) = 0\,, \end{cases} \tag{2.54}$$

is equivalent to the set

$$\begin{cases} A' + g_B\lambda'^B = (S' + g_A f'^A, B' + g_C\rho'^C) + \dfrac{\partial^R}{\partial \xi^C}(S' + g_A f'^A) g^C_D \rho'^C\,, \\ +(-)^A f^A \dfrac{\partial^L}{\partial \xi^A}(B' + g_C\rho'^C)\,, \\ \xi^*_A g^A_B\lambda'^B = \xi^*_A \dfrac{\partial^R g^A_B f'^B}{\partial \xi^C} g^C_D \rho'^D + (-)^D g^D_C f'^C \dfrac{\partial^L}{\partial \xi^D}(\xi^*_A g^A_B \rho'^B)\,. \end{cases} \tag{2.55}$$

JHEP03(1999)010

Using (2.51), the last equations in (2.54) and (2.55) are just the s_Q cocycle and coboundary conditions, expressed in terms of the $\xi^{*\prime}, \xi'$ variables. Following the same reasoning as in the proof of theorem 3, we get,

$$(S' + g_A f'^A, A') + \Delta_c A' = 0 \Longleftrightarrow A' = (S' + g_A f'^A, B' + g_C \rho'^C) + \\ + \Delta_c(B' + g_C \rho'^C) + \frac{\partial^R}{\partial \xi'^C}(S' + g_A f'^A)\rho'^C\,, \tag{2.56}$$

$$(\xi^{*\prime}{}_A f'^A, \xi^{*\prime}{}_B \rho'^B)_{\xi'} = 0\,, \tag{2.57}$$

where $\frac{\partial^R}{\partial \xi'^C}(S' + g_A f'^A)\rho'^C$ is $(S' + g_A f'^A, \cdot) + \Delta_c$ exact iff $\xi^{*\prime}{}_B \rho'^B = (\xi^{*\prime}{}_A f'^A, \xi^{*\prime}{}_C \nu'^C)_{\xi'}$. Since $(S' + g_A f'^A, \cdot) + \Delta_c = (S' + g_A f'^A, \cdot)_{z'} + \Delta'_c = \bar{s}'$, we get that $H(\bar{s}', F')$ is determined by $\frac{\partial^R}{\partial \xi'^C}(S' + g_A f'^A)\rho'^C$ corresponding to the class $\frac{\partial^R S}{\partial \xi^C}\rho^C$ of $H(\bar{s}, F)$. $\square$

Remark. Note that one can prove in the same way that the relations $\frac{1}{2}(A, A) + \Delta_c A = C$ and $(A, D) + \Delta_c D = E$ become, after the change of variables, $\frac{1}{2}(A' + g_A f'^A, A' + g_A f'^A) + \Delta_c(A' + g_A f'^A) = C'$, respectively $(A' + g_A f'^A, D') + \Delta_c D' = E'$.

3. Renormalization

We show how the renormalization can be performed while respecting as much as possibe the symmetry in the form of the extended master equation: the corresponding extended Zinn-Justin equation for the renormalized effective action is shown to be broken only by non trivial anomalies.

3.1 Regularization

We apply the discussion and notations of dimensional regularization given in [32] to the extended master equation (2.38) and its solution $S(\xi)$[6]. In the following we will always understand the ξ dependence without explicitly indicating it. Local functionals are understood to belong to F. In fact, we will only use the following three properties of dimensional regularization [32]:

- the regularized action $S_\tau = \sum_{n=0} \tau^n S_n$ is a polynomial or a power series in τ, the classical starting point action S corresponding to S_0,[7]
- if the renormalization has been carried out to $n-1$ loops, the divergences of the effective action at n loops are poles in τ up to the order n with residues that are local functionals, and

[6] It is also possible to apply directly the rigorously proved renormalized quantum action principles [43] (proved in the context of dimensional renormalization in [33], see also [34]). We choose not to do so here, because we want to keep the divergent counterterms explicitly in the discussion, to allow on the one hand for a direct comparison with the discussions in [20, 21] and, on the other hand, to be able to define the Batalin-Vilkovisky Δ operator in this context.

[7] We assume that all the algebraic relations holding for the classical action S hold in the regularized theory for S_0.

JHEP03(1999)010

- the regularized quantum action principle holds (see the first reference of [33], sections II.3 and II.4).

Let $\theta_\tau = \frac{1}{2\tau}(S_\tau, S_\tau) + \frac{1}{\tau}\Delta_c S_\tau$. Note that θ_τ is of order τ^0 because S_0 satisfies the extended master equation. θ_τ characterizes the breaking of the extended master equation due to the regularization. In order to control this breaking during renormalization, it is useful to couple it with a global source ρ^* in ghost number -1 and consider $S_{\rho^*} = S_\tau + \theta_\tau \rho^*$. On the classical, regularized level, we have, using $(\rho^*)^2 = 0$, and the properties (2.36) and (2.37) of Δ_c,

$$\frac{1}{2}(S_{\rho^*}, S_{\rho^*}) + \Delta_c S_{\rho^*} = \tau \frac{\partial^R S_{\rho^*}}{\partial \rho^*}\,, \tag{3.1}$$

Applying the quantum action principle, we get, for the regularized generating functional for 1PI irreducible vertex functions Γ_{ρ^*} associated to S_{ρ^*},

$$\frac{1}{2}(\Gamma_{\rho^*}, \Gamma_{\rho^*}) + \Delta_c \Gamma_{\rho^*} = \tau \frac{\partial^R \Gamma_{\rho^*}}{\partial \rho^*}\,, \tag{3.2}$$

which splits, using $(\rho^*)^2 = 0$, into

$$\frac{1}{2}(\Gamma, \Gamma) + \Delta_c \Gamma = \tau \frac{\partial^R \Gamma_{\rho^*}}{\partial \rho^*}\,, \tag{3.3}$$

$$\left(\Gamma, \frac{\partial^R \Gamma_{\rho^*}}{\partial \rho^*}\right) + \Delta_c \frac{\partial^R \Gamma_{\rho^*}}{\partial \rho^*} = 0\,. \tag{3.4}$$

3.2 Invariant regularization

Before proceeding with the general analysis, let us briefly discuss the case when dimensional regularization is an invariant regularization scheme for the symmetries under considerations. (The arguments below can be adapted in a straightforward way to other invariant regularization schemes).

In this case, θ_τ vanishes and we have

$$\frac{1}{2}(S_\tau, S_\tau) + \Delta_c S_\tau = 0\,. \tag{3.5}$$

For the regularized generating functional, we get

$$\frac{1}{2}(\Gamma, \Gamma) + \Delta_c \Gamma = 0\,, \tag{3.6}$$

where by assumption, $\Gamma = S_\tau + \hbar \sum_{n=-1} \tau^n \Gamma^{(1)n} + O(\hbar^2)$. To order $\hbar/\tau$, (3.6) gives

$$\bar{s}\Gamma^{(1)-1} = 0 \Longleftrightarrow \Gamma^{(1)-1} = \bar{s}\Xi_1 + \frac{\partial^R S_0}{\partial \xi^A}\mu_1^A, \tag{3.7}$$

where $\bar{s} = (S_0, \cdot) + \Delta_c$ and $s_Q\ \xi^*_A \mu_1^A = 0$.

JHEP03(1999)010

We then make the following change of fields, antifields and coupling constants:

$$z^1 = \exp -\frac{\hbar}{\tau}(\cdot, \Xi_1 + \xi^*\mu_1\tilde{)} z \,, \tag{3.8}$$

$$\xi^1 = \exp -\frac{\hbar}{\tau}(\cdot, \xi^*\mu_1)_\xi \xi \,. \tag{3.9}$$

If we denote by a superscript 1 functions depending on these new variables, we have, according to the remark after theorem 4, that the action $S_{R1} = S^1_\tau + g_{1\,A} f^{1^A}$, where $g_{1\,A}$ is determined through the generators Ξ_1 and μ_1^C of the first redefinition, satisfies the extended master equation (2.38),

$$\frac{1}{2}(S_{R_1}, S_{R_1}) + \Delta_c S_{R_1} = 0 \,, \tag{3.10}$$

and allows to absorb the one loop divergences, since $S_{R1} = S_\tau - \hbar/\tau \Gamma^{(1)-1} + \hbar O(\tau^0) + O(\hbar^2)$. We thus have for the corresponding regularized generating functional $\Gamma_{R_1} = S_\tau + \hbar \sum_{n=0} \tau^n \Gamma^{(1)n}_{R_1} + \hbar^2 \sum_{n=-2} \tau^n \Gamma^{(2)n}_{R_1} + O(\hbar^2)$,

$$\frac{1}{2}(\Gamma_{R_1}, \Gamma_{R_1}) + \Delta_c \Gamma_{R_1} = 0 \,. \tag{3.11}$$

At order $\hbar^2/\tau^2$, we get

$$\bar{s}\Gamma^{(2)-2}_{R_1} = 0 \Longleftrightarrow \Gamma^{(2)-2} = \bar{s}\Xi_{2,-2} + \frac{\partial^R S_0}{\partial \xi^A}\mu^A_{2,-2} \,, \tag{3.12}$$

with $s_Q\ \xi^*_A \mu^A_{2,-2} = 0$. The appropriate change of variables is

$$z^{2,-2} = \exp -\frac{\hbar^2}{\tau^2}(\cdot, \Xi_{2,-2} + \xi^*\mu_{2,-2}\tilde{)} z \,, \tag{3.13}$$

$$\xi^{2,-2} = \exp -\frac{\hbar^2}{\tau^2}(\cdot, \xi^*\mu_{2,-2})_\xi \xi \,. \tag{3.14}$$

The regularized action $S_{R_{2,-2}} = S^{2,-2}_{R_1} + g_{2,-2\,A} f^{2,-2\,A}$ satisfies the extended master equation and allows to absorb the poles of order $\hbar^2/\tau^2$:

$$\frac{1}{2}(S_{R_{2,-2}}, S_{R_{2,-2}}) + \Delta_c S_{R_{2,-2}} = 0 \,, \tag{3.15}$$

and $\Gamma_{R_{2,-2}} = S_\tau + \hbar \sum_{n=0} \Gamma^{(1)n}_{R_{2,-2}} + \hbar^2 \sum_{n=-1} \tau^n \Gamma^{(2)n}_{R_{2,-2}} + O(\hbar^3)$.

In the same way, one can then proceed to absorb the poles of order $\hbar^2/\tau$ to get a regularized action $S_{R_{2,-1}}$ and an associated two loop finite effective action $\Gamma_{R_{2,-1}}$, with both actions satisfying the extended master equation.

Going on recursively to higher orders in $\hbar$, we can achieve, through a succession of redefinitions, the absorptions of the infinities to arbitrary high order in the loop expansion, while preserving the extended master equation for the redefined action and the corresponding generating functional.

JHEP03(1999)010

[Symbolically,

$$\frac{1}{2}(S_{R_\infty}, S_{R_\infty}) + \Delta_c S_{R_\infty} = 0\,, \tag{3.16}$$

with Γ_{R_∞} finite and satisfying

$$\frac{1}{2}(\Gamma_{R_\infty}, \Gamma_{R_\infty}) + \Delta_c \Gamma_{R_\infty} = 0\,.\,\Big] \tag{3.17}$$

We have thus shown:

Theorem 5 *In theories admitting an invariant regularization scheme, the divergences can be absorbed by successive redefinitions in such a way that both the subtracted and the effective action satisfy the extended master equation.*

3.3 Structural constraints and cohomology of $\bar{s}$

Structural constraints have been introduced in [21] to give in particular cases a sufficient, but not a necessary condition for renormalizability in the modern sense. In the cases of semi-simple Yang-Mills theories or gravity for instance, these constraints correspond to the prescription of a choice for the representatives of the BRST cohomology classes in ghost number 0 (to be coupled to the bare action, if not already contained therein): the representatives should be taken to be independent of the antifields. Because one can prove that in every BRST cohomology class in ghost number 0, there exists such a representative, renormalizability in the modern sense is guaranteed to hold.

But in these examples, one expects renormalizability in the modern sense to hold, even if one chooses different representatives for the cohomology classes. Consider for instance semi-simple Yang-Mills theory. The choice of representatives in agreement with the structural constraint is to take the Yang-Mills action itself as a representative, or, in other words, to consider the coupling k in front of the Yang-Mills action as an essential one. One could also take the derivative of the solution of the master equation with respect to the coupling constant g associated with the structure constant as a representative for this cohomology class, and this representative depends on the antifields. That these two terms are in the same cohomology class follows from the well-known field redefinitions that allow to absorb either one of them. In other words, only one of the couplings k or g is an essential one. Choosing g as an essential coupling does not respect the structural constraint, but clearly, one does not expect the validity of renormalizability in the modern sense to depend on this choice.

What has been shown in the previous sections is that renormalizability in the modern sense does not depend on how one chooses the representatives, or equivalently, the essential couplings, and that structural constraints are not necesseary conditions for renormalizability in the modern sense. This has been done by taking into

JHEP03(1999)010

account higher order cohomology restrictions, incorporated in the extended antifield formalism through the cohomology of the operator $\bar{s}$. As shown in [21], structural constraints are nevertheless very useful in concrete cases, to show renormalizability in the modern sense, without using the more heavy machinery developped here, which consists in controling the renormalization of the complete theory with all its generalized observables.

In the non anomalous case, the extended master equation for the effective action implies a remarkable stability of the quantum theory: while the expression of the generalized observables of the theory are affected by quantum corrections, their antibracket algebra stays the same than in the classical theory. In particular, the usual algebra of the generators of the global symmetries (whether linear or not) is the same in the classical and the quantum theory.[8] This is because the antibracket algebra of the BRST cohomology classes in negative ghost numbers just reflects the ordinary algebra of the symmetries they represent.

3.4 One loop divergences and anomalies

Let us now go back to the general case where the dimensional regularization scheme is not invariant.[9]

At one loop, we get from (3.3) and (3.4)

$$(S_\tau, \Gamma^{(1)}) + \Delta_c \Gamma^{(1)} = \tau\theta^{(1)}\,, \tag{3.18}$$

$$(S_\tau, \theta^{(1)}) + (\Gamma^{(1)}, \theta_\tau) + \Delta_c \theta^{(1)} = 0\,, \tag{3.19}$$

where $\Gamma^{(1)}$ and $\theta^{(1)}$ are respectively the one loop contributions of Γ and $\frac{\partial^R \Gamma_{\rho^*}}{\partial \rho^*}$. By assumption, we have both $\Gamma^{(1)} = \sum_{n=-1} \tau^n \Gamma^{(1)n}$ and $\theta^{(1)} = \sum_{n=-1} \tau^n \theta^{(1)n}$, where $\Gamma^{(1)-1}, \theta^{(1)-1}$ are local functionals.

At $\frac{1}{\tau}$, equation (3.18) gives

$$\bar{s}\Gamma^{(1)-1} = 0\,, \tag{3.20}$$

Using this equation together with $\theta_0 = \bar{s}S_1$, equation (3.19) implies

$$\bar{s}\left(\theta^{(1)-1} - (\Gamma^{(1)-1}, S_1)\right) = 0\,. \tag{3.21}$$

Equation (3.18) also gives at order τ^0

$$\bar{s}\Gamma^{(1)0} = \theta^{(1)-1} - (\Gamma^{(1)-1}, S_1)\,, \tag{3.22}$$

which allows us to identify the combination $A_1 = \theta^{(1)-1} - (\Gamma^{(1)-1}, S_1)$ as the one loop anomaly and explicitly shows its locality. We have thus shown in the case of a non invariant regularization scheme:

Theorem 6 *The one loop divergences $\Gamma^{(1)-1}$ and the one loop anomalies A_1 are $\bar{s}$ cocycles in ghost number 0 and 1 respectively.*

[8] The author is grateful to F. Brandt for pointing this out.

[9] The derivation of some of the results below in the framework of algebraic renormalization [36] will be discussed elsewhere [44].

JHEP03(1999)010

3.5 One loop renormalization

According to (2.31), we have

$$\Gamma^{(1)-1} = \bar{s}\Xi_1 + \frac{\partial^R S_0}{\partial \xi^D}\mu_1^D \tag{3.23}$$

and

$$A_1 = \bar{s}\Sigma_1 + \frac{\partial^R S_0}{\partial \xi^E}\sigma_1^E\,, \tag{3.24}$$

with $s_Q\ \xi^*_A\mu_1^A = 0 = s_Q\xi^*_B\sigma_1^B$. The appropriate change of variables is now

$$z^1 = \exp -\frac{\hbar}{\tau}\widetilde{(\cdot, \Xi_1 + \xi^*\mu_1)} z\,, \tag{3.25}$$

$$\xi^1 = \exp -\frac{\hbar}{\tau}\widetilde{(\cdot, \xi^*\mu_1)}\xi\,. \tag{3.26}$$

The renormalized one loop action is

$$S_{R_1} = S^1_\tau + g_{1A}f^{1A} - \hbar\tilde{\Sigma}^1_1 = S_\tau - \frac{\hbar}{\tau}\Gamma^{(1)-1} + \hbar O(\tau^0) + O(\hbar^2)\,, \tag{3.27}$$

where $\tilde{\Sigma}_1$ remains to be determined. Using the remark after theorem 4, we get

$$\begin{aligned}
\theta_{R_1} &\equiv \frac{1}{2\tau}(S_{R_1}, S_{R_1}) + \frac{1}{\tau}\Delta_c S_{R_1} \\
&= \theta^1_\tau - \frac{\hbar}{\tau}(\bar{s}\tilde{\Sigma}_1)^1 + O(\hbar^2) \\
&= \theta_\tau - \frac{\hbar}{\tau}\bar{s}\left[\tilde{\Sigma}_1 + (S_1, \Xi_1) + \frac{\partial^R S_1}{\partial \xi^A}\mu_1^A\right] - \frac{\hbar}{\tau}(\Gamma^{(1)-1}, S_1) + \hbar O(\tau^0) + O(\hbar^2)\,.
\end{aligned} \tag{3.28}$$

Finally, we consider $\xi^1_{\rho^*} = \exp -\frac{\hbar}{\tau}(\cdot, \xi^*\sigma_1\rho^*)\xi = \xi - \frac{\hbar}{\tau}\sigma_1\rho^*$ and substitute ξ by $\xi^1_{\rho^*}$:

$$\begin{aligned}
S^{\rho^*}_{R_1}(z, \xi, \rho^*) &\equiv S_{R_1}(z, \xi^1_{\rho^*}(\xi, \rho^*)) \\
&= S_{R_1}(z, \xi) - \frac{\hbar}{\tau}\frac{\partial^R S_{R_1}}{\partial \xi^A}\sigma_1^A\rho^* && (3.29)\\
&= S_{R_1}(z, \xi) - \frac{\hbar}{\tau}\frac{\partial^R S_0}{\partial \xi^A}\sigma_1^A\rho^* + \hbar O(\tau^0) + O(\hbar^2)\,. && (3.30)
\end{aligned}$$

We also have that

$$\begin{aligned}
\theta^{\rho^*}_{R_1}(z, \xi, \rho^*) &\equiv \theta_{R_1}(z, \xi^1_{\rho^*}(\xi, \rho^*)) \\
&= \theta_{R_1}(z, \xi) - \frac{\hbar}{\tau}\frac{\partial^R \theta_{R_1}}{\partial \xi^A}\sigma_1^A\rho^* && (3.31)\\
&= \frac{1}{2\tau}(S^{\rho^*}_{R_1}, S^{\rho^*}_{R_1}) + \frac{1}{\tau}\Delta_c S^{\rho^*}_{R_1}\,. && (3.32)
\end{aligned}$$

JHEP03(1999)010

Equations (3.28) and (3.30) imply that the action

$$S_{R_1\rho^*} = S_{R_1}^{\rho^*} + \theta_{R_1}^{\rho^*}\rho^* , \tag{3.33}$$

with $\tilde{\Sigma}_1 = \Sigma_1 - (S_1, \Xi_1) - \frac{\partial^R S_1}{\partial \xi^A}\mu_1^A$, yields a one loop finite effective action both in the ρ^* independent and the ρ^* linear part, because the terms linear in ρ^* of order $\hbar/\tau$ add up precisely to $-\theta^{(1)-1}$. The one loop renormalized and regularized action $S_{R_1\rho^*}$ satisfies

$$\begin{aligned}\frac{1}{2}(S_{R_1\rho^*}, S_{R_1\rho^*}) + \Delta_c S_{R_1\rho^*} &= \tau\theta_{R_1}^{\rho^*} \\ &= \tau\frac{\partial^R S_{R_1\rho^*}}{\partial\rho^*} + \frac{\partial^R S_{R_1\rho^*}}{\partial\xi^B}\hbar\sigma_1^B - \frac{1}{\tau}\frac{\partial^R S_{R_1\rho^*}}{\partial\xi^B}\frac{\partial^R\hbar\sigma_1^B}{\partial\xi^A}\hbar\sigma_1^A\rho^* ,\end{aligned} \tag{3.34}$$

the first equality following from (3.32), and the last one from the expansions (3.29), (3.31), together with the identity

$$(-)^{B(A+1)}\frac{\partial^R}{\partial\xi^B}\left(\frac{\partial^R S_{R_1}}{\partial\xi^A}\right)\sigma_1^A\sigma_1^B = 0 . \tag{3.35}$$

According to the regularized quantum action principle,

$$\begin{aligned}\frac{1}{2}(\Gamma_{R_1\rho^*}, \Gamma_{R_1\rho^*}) + \Delta_c\Gamma_{R_1\rho^*} = \tau\frac{\partial^R\Gamma_{R_1\rho^*}}{\partial\rho^*} + \frac{\partial^R\Gamma_{R_1\rho^*}}{\partial\xi^B}\hbar\sigma_1^B - \\ -\frac{1}{\tau}\frac{\partial^R\Gamma_{R_1\rho^*}}{\partial\xi^B}\frac{\partial^R\hbar\sigma_1^B}{\partial\xi^A}\hbar\sigma_1^A\rho^* .\end{aligned} \tag{3.36}$$

The ρ^* independent part at one loop and lowest order, τ^0, in τ gives

$$\bar{s}\Gamma_{R_1}{}^{(1)0} = \frac{\partial^R S_0}{\partial\xi^B}\sigma_1^B , \tag{3.37}$$

and shows that only the non trivial part of the anomaly remains.

3.6 Two loops

3.6.1 Equations for the two loop poles

The one loop renormalized action admits the expansion

$$\Gamma_{R_1\rho^*} = S_{\rho^*} + \hbar\Sigma_{n=0}\tau^n\Gamma_{R_1\rho^*}^{(1)n} + \hbar^2\Sigma_{n=-2}\tau^n\Gamma_{R_1\rho^*}^{(2)n} + O(\hbar^3) . \tag{3.38}$$

At order $\hbar^2$ (3.36) gives,

$$\begin{aligned}(S_{\rho^*}, \Gamma_{R_1\rho^*}^{(2)}) + \frac{1}{2}(\Gamma_{R_1\rho^*}^{(1)}, \Gamma_{R_1\rho^*}^{(1)}) + \Delta_c\Gamma_{R_1\rho^*}^{(2)} = \\ = \tau\frac{\partial^R\Gamma_{R_1\rho^*}^{(2)}}{\partial\rho^*} + \frac{\partial^R\Gamma_{R_1\rho^*}^{(1)}}{\partial\xi^B}\sigma_1^B - \frac{1}{\tau}\frac{\partial^R S_0}{\partial\xi^B}\frac{\partial^R\sigma_1^B}{\partial\xi^A}\sigma_1^A\rho^* .\end{aligned} \tag{3.39}$$

JHEP03(1999)010

Let $\Gamma_{R_1\rho^*} = \Gamma_{R_1} + \frac{\partial^R \Gamma_{R_1\rho^*}}{\partial \rho^*}\rho^*$. At order $1/\tau^2$, we get, according to the ρ^* independent and linear parts,

$$\bar{s}\Gamma_{R_1}{}^{(2)-2} = 0\,, \tag{3.40}$$

$$\bar{s}\left(\frac{\partial^R \Gamma_{R_1\rho^*}^{(2)-2}}{\partial \rho^*} - \left(S_1, \Gamma_{R_1}{}^{(2)-2}\right)\right) = 0\,. \tag{3.41}$$

The first of these equations implies:

Lemma 2 *The second order pole of the two loop divergences is a $\bar{s}$ cocycle.*

At order $1/\tau$, we get

$$\bar{s}\Gamma_{R_1}{}^{(2)-1} = \frac{\partial^R \Gamma_{R_1\rho^*}^{(2)-2}}{\partial \rho^*} - (S_1, \Gamma_{R_1}^{(2)-2})\,, \tag{3.42}$$

$$\bar{s}\left(\frac{\partial^R \Gamma_{R_1\rho^*}^{(2)-1}}{\partial \rho^*} - (S_1, \Gamma_{R_1}{}^{(2)-1}) - (S_2, \Gamma_{R_1}^{(2)-2})\right) = -\frac{\partial^R S_0}{\partial \xi^B}\frac{\partial^R \sigma_1^B}{\partial \xi^A}\sigma_1^A\,. \tag{3.43}$$

Finally, the ρ^* independent part of (3.39), gives at order τ^0

$$\begin{aligned}\bar{s}\Gamma_{R_1}{}^{(2)0} + \frac{1}{2}(\Gamma_{R_1}{}^{(1)0}, \Gamma_{R_1}{}^{(1)0}) = {} & \frac{\partial^R \Gamma_{R_1}{}^{(1)0}}{\partial \xi^B}\sigma_1^B + \frac{\partial^R \Gamma_{R_1\rho^*}^{(2)-1}}{\partial \rho^*} - \\ & -(S_1, \Gamma_{R_1}{}^{(2)-1}) - (S_2, \Gamma_{R_1}{}^{(2)-2})\,,\end{aligned} \tag{3.44}$$

which allows to identify the combination

$$A_2 = \frac{\partial^R \Gamma_{R_1\rho^*}^{(2)-1}}{\partial \rho^*} - (S_1, \Gamma_{R_1}{}^{(2)-1}) - (S_2, \Gamma_{R_1}{}^{(2)-2}) \tag{3.45}$$

as the local contribution to the two loop anomaly, whereas $\frac{\partial^R \Gamma_{R_1}{}^{(1)0}}{\partial \xi^B}\sigma_1^B$ is the one loop renormalized dressing of the non trivial one loop anomaly.

3.6.2 Two loop anomaly consistency condition

Before absorbing the divergences, let us consider (3.43), which can be written as

$$\bar{s}A_2 = -\frac{1}{2}\frac{\partial^R S_0}{\partial \xi^B}[\sigma^1, \sigma^1]^B\,, \tag{3.46}$$

where $\xi^*_B[\sigma^1, \sigma^1]^B \equiv (\xi^*\sigma_1, \xi^*\sigma_1)_\xi$ is an s_Q cocyle because of the graded Jacobi identity for the antibracket in ξ, ξ^* space. According to equations (2.27) and (2.28), this implies that

$$\frac{1}{2}(\xi^*\sigma_1, \xi^*\sigma_1)_\xi = s_Q \xi^*_A \sigma_2^A\,, \tag{3.47}$$

and

$$A_2 = \bar{s}\Sigma_2 + \frac{\partial^R S_0}{\partial \xi^B}\sigma_2^B\,. \tag{3.48}$$

JHEP03(1999)010

Discussion. We thus see that the consistency condition (3.46) on the local contribution of the two loop anomaly does not require it to be just a cocycle of the extended BRST differential $\bar{s}$, because of the non vanishing right hand side. This is in agreement with the analysis of [28]. Nevertheless, in the extended antifield formalism, the general solution of (3.48) can be characterized: it is given by a $\bar{s}$ boundary up to the term $\frac{\partial^R S_0}{\partial \xi^B}\sigma_2^B$ with the following interpretation. From the point of view of cohomology, equation (3.46) should be understood as a restriction on the non trivial one loop anomalies $\xi^*_A\sigma_1^A$ that can arise. Indeed, its consequence is (3.47), which states that the non trivial one loop anomalies should have a trivial antibracket map[10] among themselves. This is a cohomological statement independent of the choice of representatives. The term $\frac{\partial^R S_0}{\partial \xi^B}\sigma_2^B$ of the general solution for the local part of the two loop anomaly is determined by an arbitrary s_Q cocycle up to a particular solution depending on the choice of representatives for the non trivial one loop anomalies and needed to make the bracket $(\xi^*\sigma_1, \xi^*\sigma_1)_\xi$ s_Q exact. This answers, at least in the present context of the extended antifield formalism and dimensional regularization,[11] the question raised in [28] on the cohomological interpretation of the two loop anomaly consistency condition. One also sees on this example how the discussion of the quantum Batalin-Vilkovisky formalism is shifted to ξ, ξ^* space in the extended formalism.

Note that, as in [45], this result has been achieved by adding a BRST breaking counterterm, not only for the one loop divergences produced by the standard action itself, but also for the one loop divergences produced by the insertion of the non trivial one loop anomaly. This is because this anomaly has been coupled to the action itself from the start, and the BRST breaking counterterm Σ_1 also depends on the corresponding coupling constants.

3.6.3 Two loop renormalization

The general solution to (3.40) is ${\Gamma_{R_1}}^{(2)-2} = \bar{s}\Xi_{2,-2} + \frac{\partial^R S_0}{\partial \xi^A}\mu^A_{2,-2}$. We consider the change of variables

$$z^{2,-2} = \exp -\frac{\hbar^2}{\tau^2}[(\cdot, \Xi^{\rho^*}_{2,-2}) + (\cdot, {\xi^1_{\rho^*}}^* \mu^{\rho^*}_{2,-2})_{\xi^1_{\rho^*}}]z\,, \tag{3.49}$$

$$\xi^{2,-2} = \exp -\frac{\hbar^2}{\tau^2}(\cdot, {\xi^1_{\rho^*}}^* \mu^{\rho^*}_{2,-2})_{\xi^1_{\rho^*}}\xi^1_{\rho^*}\,, \tag{3.50}$$

where $\Xi^{\rho^*}_{2,-2}(z,\xi,\rho^*) = \Xi_{2,-2}(z,\xi^1_{\rho^*}(\xi,\rho^*))$ and $\mu^{\rho^*}_{2,-2}(\xi,\rho^*) = \mu_{2,-2}(\xi^1_{\rho^*}(\xi,\rho^*))$. The fact that we consider this change of variables in terms of $\xi^1_{\rho^*}$ instead of ξ will not change the absorption of the ρ^* independent divergences, but it will be important in order to control the dependence on ρ^* below. Equation (3.42) means that there is

[10]The antibracket map here is the antibracket induced in the s_Q cohomology classes from the antibracket in ξ space. Its construction can be obtained by just following the construction of the usual antibracket map used in [23, 24].

[11]The analysis of [44] confirms these results in the context of algebraic renormalization.

JHEP03(1999)010

no non trivial part $\frac{\partial^R S_0}{\partial \xi^A}\sigma^A_{2,-2}$ in the general solution to (3.41) and hence no need for a renormalization of the coupling constants of order $\hbar^2/\tau^2$ proportional to ρ^*. The general solution to (3.41) is $\frac{\partial^R \Gamma_{R_1}{}_{\rho^*}^{(2)-2}}{\partial \rho^*} - (S_1, \Gamma_{R_1}{}^{(2)-2}) = \bar{s}\Sigma_{2,-2}$, where $\Sigma_{2,-2}$ can be identified with a particular solution $\Gamma_{R_1}{}_P^{(2)-1}$ of (3.42). We take

$$\begin{aligned} S^{\rho^*}_{R_{2,-2}}(z,\xi,\rho^*) &= S_{R_1}(z^{2,-2}(z,\xi^1_{\rho^*}),\xi^{2,-2}(\xi^1_{\rho^*})) + g_{2,-2A}(z,\xi^1_{\rho^*})f^A(\xi^1_{\rho^*}) - \\ &\quad -\frac{\hbar^2}{\tau}\tilde{\Sigma}_{2,-2}(z^{2,-2}(z,\xi^1_{\rho^*}),\xi^{2,-2}(\xi^1_{\rho^*})) \\ &= S_{R_1} - \frac{\hbar^2}{\tau^2}\Gamma_{R_1}{}^{(2)-2} + O(\hbar^2\tau^{-1}) + O(\hbar^3)\,, \end{aligned} \tag{3.51}$$

where $\tilde{\Sigma}_{2,-2}(z,\xi)$ remains to be determined. The remark after theorem 4 again implies

$$\begin{aligned} \theta^{\rho^*}_{R_{2,-2}} &\equiv \frac{1}{2\tau}(S^{\rho^*}_{R_{2,-2}}, S^{\rho^*}_{R_{2,-2}}) + \frac{1}{\tau}\Delta_c S^{\rho^*}_{R_{2,-2}} \\ &= \theta^{\rho^*}_{R_1} - \frac{\hbar^2}{\tau^2}\bar{s}\left[\tilde{\Sigma}_{2,-2} + \frac{\partial^R S_1}{\partial \xi^A}\mu^A_{2,-2} + (S_1,\Xi_{2,-2})\right] - \\ &\quad -\frac{\hbar^2}{\tau^2}(S_1, \Gamma_{R_1}{}^{(2)-2}) + \hbar^2 O(\tau^{-1}) + O(\hbar^3)\,. \end{aligned} \tag{3.52}$$

The action

$$S_{R_{2,-2}\rho^*} = S^{\rho^*}_{R_{2,-2}} + \theta^{\rho^*}_{R_{2,-2}}\rho^*\,, \tag{3.53}$$

with $\tilde{\Sigma}_{2,-2} = \Sigma_{2,-2} - \frac{\partial^R S_1}{\partial \xi^A}\mu^A_{2,-2} + (S_1,\Xi_{2,-2})$, yields an effective action $\Gamma_{R_{2,-2}\rho^*}$ without $\hbar^2/\tau^2$ divergences and only simple poles at order $\hbar^2$, because the terms linear in ρ^* of order $\hbar^2/\tau^2$ add up precisely to $-\frac{\partial^R \Gamma_{R_1}{}_{\rho^*}^{(2)-2}}{\partial \rho^*}$. We have again that

$$\begin{aligned} \frac{1}{2}(S_{R_{2,-2}\rho^*}, S_{R_{2,-2}\rho^*}) + \Delta_c S_{R_{2,-2}\rho^*} &= \tau\theta^{\rho^*}_{R_{2,-2}} = \\ &= \tau\frac{\partial^R S_{R_{2,-2}\rho^*}}{\partial \rho^*} + \frac{\partial^R S_{R_{2,-2}\rho^*}}{\partial \xi^B}\hbar\sigma^B_1 - \frac{1}{\tau}\frac{\partial^R S_{R_{2,-2}\rho^*}}{\partial \xi^B}\frac{1}{2}[\hbar\sigma_1,\hbar\sigma_1]^B\rho^*\,. \end{aligned} \tag{3.54}$$

The last equation follows from the fact that the dependence of $S^{\rho^*}_{R_{2,-2}}$ and $\theta^{\rho^*}_{R_{2,-2}}$ on ρ^* is, as before, through the combination $\xi^1_{\rho^*}$. The same equation holds again for the effective action:

$$\begin{aligned} \frac{1}{2}(\Gamma_{R_{2,-2}\rho^*}, \Gamma_{R_{2,-2}\rho^*}) + \Delta_c \Gamma_{R_{2,-2}\rho^*} &= \tau\frac{\partial^R \Gamma_{R_{2,-2}\rho^*}}{\partial \rho^*} + \frac{\partial^R \Gamma_{R_{2,-2}\rho^*}}{\partial \xi^B}\hbar\sigma^B_1 - \\ &\quad -\frac{1}{\tau}\frac{\partial^R \Gamma_{R_{2,-2}\rho^*}}{\partial \xi^B}\frac{1}{2}[\hbar\sigma_1,\hbar\sigma_1]^B\rho^*\,. \end{aligned} \tag{3.55}$$

The expansion of this effective action is

$$\Gamma_{R_{2,-2}\rho^*} = S_{\rho^*} + \hbar\Sigma_{n=0}\tau^n\Gamma_{R_{2,-2}}{}_{\rho^*}^{(1)n} + \hbar^2\Sigma_{n=-1}\tau^n\Gamma_{R_{2,-2}}{}_{\rho^*}^{(2)n} + O(\hbar^3)\,. \tag{3.56}$$

JHEP03(1999)010

The divergences $\Gamma^{(2)-1}_{R_{2,-2}}$ and $\frac{\partial^R \Gamma_{R_{2,-2}}{}^{(2)-1}_{\rho^*}}{\partial \rho^*}$, now satisfy $\bar{s}\Gamma^{(2)-1}_{R_{2,-2}} = 0$ and $\bar{s}A'_2 = \frac{\partial^R S_0}{\partial \xi^A}\frac{1}{2}[\sigma^1, \sigma^1]^A$, with $A'_2 = \frac{\partial^R \Gamma_{R_{2,-2}}{}^{(2)-1}_{\rho^*}}{\partial \rho^*} - (\Gamma^{(2)-1}_{R_{2,-2}}, S_1)$. The general solutions are $\Gamma^{(2)-1}_{R_{2,-2}} = \bar{s}\Xi_{2,-1} + \frac{\partial^R S_0}{\partial \xi^A}\mu_{2,-1}$ and $A'_2 = \bar{s}\Sigma_{2,-1} + \frac{\partial^R S_0}{\partial \xi^A}\sigma^A_2$. As in the one loop case, one first subtracts a suitably defined BRST breaking counterterm, then one makes the field-antifield and coupling constant redefinition determined by $\Xi^{\rho^*}_{2,-1}$ and $\mu^{\rho^*}_{2,-1}$, and finally, one substitues $\xi^1_{\rho^*}$ everywhere by $\xi^2_{\rho^*} = \xi^1_{\rho^*} - \frac{\hbar^2}{\tau}\sigma_2\rho^*$, giving a total ρ^* dependence through the combination $\xi^2_{\rho^*} = \xi - \frac{\hbar}{\tau}\sigma_1\rho^* - \frac{\hbar^2}{\tau}\sigma_2\rho^*$.

Using the same arguments as in the one loop case, one finally finds that the two loop renormalized and regularized action $S_{R_2\rho^*}$ satisfies

$$\frac{1}{2}(S_{R_2\rho^*}, S_{R_2\rho^*}) + \Delta_c S_{R_2\rho^*} = \tau\frac{\partial^R S_{R_2\rho^*}}{\partial \rho^*} + \frac{\partial^R S_{R_2\rho^*}}{\partial \xi^B}(\hbar\sigma^B_1 + \hbar^2\sigma^B_2) - \\ -\frac{1}{\tau}\frac{\partial^R S_{R_2\rho^*}}{\partial \xi^B}\frac{1}{2}[\hbar\sigma_1 + \hbar^2\sigma_2, \hbar\sigma_1 + \hbar^2\sigma_2]^B\rho^*\,, \quad (3.57)$$

the same equation holding for the two loop renormalized effective action $\Gamma_{R_2\rho^*}$.

3.7 Higher orders

It is then possible to continue recursively to higher loops to get a completely subtracted and regularized action S_{R_∞}. It is obtained from

$$S_\tau - \sum_{n=1}\frac{\hbar^n}{\tau^{n-1}}\sum_{k=0}^{n-1}\tau^k\tilde{\Sigma}_{n,k-n}\,, \quad (3.58)$$

with suitably choosen BRST breaking counterterms $\tilde{\Sigma}_{n,k-n}$, by successive canonical field-antifield and coupling constants redefinitions. It satisfies

$$\frac{1}{2}(S_{R_\infty\rho^*}, S_{R_\infty\rho^*}) + \Delta_c S_{R_\infty\rho^*} = \tau\frac{\partial^R S_{R_\infty\rho^*}}{\partial \rho^*} + \frac{\partial^R S_{R_\infty\rho^*}}{\partial \xi^B}\sigma^B - \\ -\frac{1}{\tau}\frac{\partial^R S_{R_\infty\rho^*}}{\partial \xi^B}\frac{1}{2}[\sigma, \sigma]^B\rho^*\,. \quad (3.59)$$

The corresponding completely renormalized and regularized effective action $\Gamma_{R_\infty\rho^*}$ satisfies the same equation

$$\frac{1}{2}(\Gamma_{R_\infty\rho^*}, \Gamma_{R_\infty\rho^*}) + \Delta_c \Gamma_{R_\infty\rho^*} = \tau\frac{\partial^R \Gamma_{R_\infty\rho^*}}{\partial \rho^*} + \frac{\partial^R \Gamma_{R_\infty\rho^*}}{\partial \xi^B}\sigma^B - \\ -\frac{1}{\tau}\frac{\partial^R \Gamma_{R_\infty\rho^*}}{\partial \xi^B}\frac{1}{2}[\sigma, \sigma]^B\rho^*\,. \quad (3.60)$$

One can then put $\rho^* = 0$ and take safely the limit $\tau \longrightarrow 0$, because there are no more divergences left. The renormalized effective action $\Gamma^R = \lim_{\tau\to 0}\Gamma_{R_\infty\rho^*}|_{\rho^*=0}$ satisfies

$$\frac{1}{2}(\Gamma^R, \Gamma^R) + \Delta_c\Gamma^R = \frac{\partial^R \Gamma^R}{\partial \xi^B}\sigma^B\,. \quad (3.61)$$

JHEP03(1999)010

Theorem 7 *The absorption of the divergences of a theory in dimensional renormalization involves, besides redefinitions of the solution of the extended master equation, determined by anticanonical field-antifield and coupling constant renormalizations, only the subtraction of suitably chosen BRST breaking counterterms. The renormalization can be done in such a way that the anomalous breaking of the extended Zinn-Justin equation is determined to all orders by the cohomology $H^1(\bar{s}, F)$.*

The corresponding result for the anomalies in the standard Batalin-Vilkovisky formalism has been obtained in the framework of algebraic renormalization in [45].

3.8 The quantum Batalin-Vilkovisky Δ operator

In [25, 28], explicit expression for the Δ operator have been obtained in the context of Pauli-Villars and non local regularization respectively. The aim of this section is to get such an expression in the context of dimensional renormalization. The expression we will get here will be defined on all the generalized observables of the theory, and not only on S alone, since they are contained in the solution $S(\xi)$ of the extended master equation.

As discussed for instance in section 4 of [27] in the context of the BPHZ renormalized antifield formalism, even though there is a well defined expression for the anomaly, there is no room for the formal Batalin-Vilkovisky Δ operator in the final renormalized theory. Contact with the quantum Batalin-Vilkovisky formalism in the present set-up has thus to be done on the renormalized theory before the regulator τ is removed. Moreover, as in the previous discussion of the renormalization, it turns out to be important not to put to zero the fermionic variable ρ^*, which couples the breaking of the extended master equation due to the regularization. Let us introduce the notation $W = S_{R_\infty \rho^*}$ for the completely renormalized and regularized action and $\mathcal{A} = \frac{\partial^R W}{\partial \xi^B}(\sigma^B - \frac{1}{2\tau}[\sigma,\sigma]^B \rho^*)$ so that (3.60) can be written,

$$\frac{1}{2}(\Gamma_{R_\infty \rho^*}, \Gamma_{R_\infty \rho^*}) + \Delta_c \Gamma_{R_\infty \rho^*} + \tau \frac{\partial^L \Gamma_{R_\infty \rho^*}}{\partial \rho^*} = \mathcal{A} \circ \Gamma_{R_\infty \rho^*}\,, \tag{3.62}$$

while (3.59) becomes,

$$\frac{1}{2}(W,W) + \Delta_c W + \tau \frac{\partial^L W}{\partial \rho^*} = \mathcal{A}\,. \tag{3.63}$$

Let us define $\Delta_d = \frac{\tau}{-i\hbar}\frac{\partial^L}{\partial \rho^*}$, so that we can write the above equation as

$$\frac{1}{2}(W,W) + \Delta_c W - i\hbar \Delta_d W = \mathcal{A}\,. \tag{3.64}$$

The operator Δ_d is of ghost number 1, it is nilpotent, $\Delta_d^2 = 0$, it anticommutes with Δ_c, $\{\Delta_c, \Delta_d\} = 0$, and it is a graded derivation of the antibracket, i.e. it satisfies equation (2.37) (with Δ_c replaced by Δ_d). Using the properties of the antibracket,

JHEP03(1999)010

Δ_c and Δ_d, it follows that, by applying $(W,\cdot)+\Delta_c-i\hbar\Delta_d$ to (3.64), the left hand side vanshes identically. This gives the consistency condition

$$(W,\mathcal{A})+\Delta_c\mathcal{A}-i\hbar\Delta_d\mathcal{A}=0\,. \tag{3.65}$$

Discussion. Starting from the path integral expression

$$Z(J,\phi^*,\xi,\rho^*)=\int\mathcal{D}\phi\exp\left(\frac{i}{\hbar}\left[W+\int d^nx\ J_A\phi^A\right]\right), \tag{3.66}$$

with associated effective action $\Gamma_{R_\infty\rho^*}$, standard formal path integral manipulations using integrations by parts give

$$\frac{1}{2}(\Gamma_{R_\infty\rho^*},\Gamma_{R_\infty\rho^*})+\Delta_c\Gamma_{R_\infty\rho^*}=\mathcal{A}'\circ\Gamma_{R_\infty\rho^*}\,, \tag{3.67}$$

where

$$\mathcal{A}'=\frac{1}{2}(W,W)+\Delta_cW-i\hbar''\Delta W''. \tag{3.68}$$

This expression involves $\Delta=(-)^{A+1}\frac{\delta^R}{\delta\phi^A(x)}\frac{\delta^R}{\delta\phi^*_A(x)}$, the second order functional derivative operator. The quotation marks mean that the above definition of Δ cannot be used since Δ is ill defined when acting on local functionals and thus on W. Using (3.62) for the left hand side, we get $\mathcal{A}'=-\tau\frac{\partial^LW}{\partial\rho^*}+\mathcal{A}$. Using furthermore (3.63), it follows that $-i\hbar''\Delta W''=0$, as was to be expected in dimensional regularization, where $''\delta(0)''=0$.

In equation (3.63), obtained by an analysis of the renormalization procedure, there appears the operator Δ_d, which is unexpected from the point of view of formal path integral manipulations, not taking the regularization and renormalization into account. Furthermore, the operator Δ_d has the same algebraic properties as the formal operator Δ, when acting on local functionals. In dimensional regularization, one has traded the operator Δ, vanishing on local functionals, for the operator Δ_d. We thus find, in the context of dimensional regularization, that the role of the Batalin-Vilkovisky Δ operator is played by the operator Δ_d, introduced originally in the last reference of [31].

Furthermore, (3.63) suggests that the operator Δ_c, can be understood as a classical part of the Batalin-Vilkovisky Δ operator in the extended antifield formalism. This interpretation is supported by the fact that both Δ_c and Δ_d arise in a similar way from an extended action satisfying a standard master equation in an extended space with an enlarged bracket: this was shown for Δ_c in section 2.4. In [32] in the context of the standard Batalin-Vilkovisky formalism, it was shown that Δ_d also arises from an "improved" classical master equation, if the space of fields and antifields is enlarged to enclude the global pair of variables ρ,ρ^*, the antibracket is extended to this pair and the regularized action is extended to $S_\tau+\theta_\tau\rho^*+\tau\rho$.

It might be worthwhile to point out that the there are two other "extended master equations" which can be understood in this way.

JHEP03(1999)010

(i) The famous tree level Slavnov-Taylor identity in Yang-Mills theory, gauge fixed with the help of the auxiliary B field [2, 4], can be obtained from the antifield formalism (before gauge fixation) by adding to the minimal solution of the master equation the term $\int d^n x \bar{C}^*_a B^a$ and extending the antibracket to the pair $\bar{C}^a, B^a$ and their antifields.

(ii) The extended Slavnonv-Taylor identity [46] including the BRS doublet α and χ introduced in order to control the dependence of the theory on the parameter α of covariant linear gauges can be obtained in the antifield formalism by introducing the global pair α, χ and their antifields and adding the term $\chi^*\alpha$ to the solution of the master equation [47].

4. Conclusion

In order to study the renormalization of a theory together with all its non trivial observables (described in the Batalin-Vilkovisky formalism by the BRST cohomology classes), it is natural [6] to couple these observables (more precisely, those that are not already present in the solution to the master equation coupled through essential coupling constants) with the help of new coupling constants. Considering such an action as a starting point for renormalization theory allows on the one hand to get information on the renormalization of the operators, but also [24] constraints on the divergences of the starting point theory without the additional operators. The action plus the additional observables satisfies the master equation in general only up to second order in the new coupling constants since the cohomology classes may depend on the antifields.

In this paper, we have constructed an extension of the standard solution satisfying an extended master equation. This extended master equation governs not only the symmetries of the starting point action, but also the antibracket algebra of the observables. It is the right tool to study renormalizability of a theory compatible with symmetries for the following reasons:

(i) the divergences of the theory are constrained by the differential associated to this extended master equation,

(ii) the absorption of these divergences can be performed order by order through redefinitions of the extended action determined by field-antifield and coupling constant redefinitions in such a way that the redefined action still satisfies the extended master equation, up to breakings due to the non invariant terms $\sum_{n=1} \tau^n S_n$ determined by the regularization and associated symmetry breaking counterterms $\Sigma_{n,k-n}$,

(iii) the associated renormalized effective action satisfies the extended master equation up to non trivial anomalous breakings determined by the cohomology of the extended BRST differential in ghost number 1.

JHEP03(1999)010

If one adapts the terminology of [21] and defines a theory to be renormalizable in the modern sense if properties (ii) and (iii) hold, our results mean that all theories are renormalizable in this sense.

The approach proposed in this paper is not completely formal, since on the one hand, the higher order maps can be computed in principle in a straightforward way once the local BRST cohomology groups of the theory are known, and on the other hand, there has been considerable progress in computing these groups for various models such as Yang-Mills and Chern-Simons theory [7], gravity [48, 49], p-form gauge theories [50], $N = 1$ supergravity [51] or D-strings [52].

For a theory, where the BRST cohomology and the higher order maps are completely known, the only remaining problem consists in computing the s_Q cohomology.

Acknowledgments

The author thanks F. Brandt, C. Schomblond, J. Gomis, P.A. Grassi, M. Henneaux, T. Hurth, J. Stasheff, M. Tytgat and A. Wilch for useful discussions.

A. Elimination of antifield dependent counterterms in Yang-Mills theories with $U(1)$ factors

In this appendix, we will discuss the elimination by higher order cohomological restrictions of a type of antifield dependent counterterms arising in non semi-simple Yang-Mills theories. They have been discussed for the first time in detail in [8], were analyzed from a cohomological point of view in [7] and reconsidered in the concrete context of the standard model in [53]. These counterterms (or instabilities in the terminology of [8, 53]) have the following general structure [7]:

$$K' = f^{\Delta}_{\alpha} \int d^n x \; j^{\mu}_{\Delta} A^{\alpha}_{\mu} + (A^{*\mu}_{a} X^{a}_{\mu\Delta} + y^{*}_{i} X^{i}_{\Delta}) C^{\alpha} \,,$$

where f^{Δ}_{α} are constants, A^{α}_{μ} abelian gauge fields, j^{μ}_{Δ} non trivial conserved currents and $\delta_{\Delta} A^{a}_{\mu} = X^{a}_{\mu\Delta}, \delta_{\Delta} y^{i} = X^{i}_{\Delta}$ the generators of the corresponding symmetries on all the gauge fields A^{a}_{μ} and the matter fields y^{i}. In order to eliminate these instabilities by cohomological means, we will show that:

it is sufficient that there exists a set of local, non integrated, off-shell gauge invariant polynomials $O_{\Gamma}(x)$ constructed out of the A^{a}_{μ}, y^{i} and their derivatives, that break the global symmetries δ_{Δ} in the following sense: the variation of $O_{\Gamma}(x)$ under the gauged global symmetries δ_{Δ} with gauge parameter given by $f^{\Delta}_{\alpha}\epsilon^{\alpha}$ should not be equal on shell to an ordinary gauge transformation (involving the abelian gauge parameters ϵ^{α} alone) of some local polynomials $P_{\Gamma}(x)$ constructed out of the A^{a}_{μ}, y^{i} and their derivatives.

JHEP03(1999)010

Indeed, using the extended action[12] $S_{k(x)} = S + \int d^n x\, k^\Gamma(x) O_\Gamma(x)$, which satisfies $1/2(S_{k(x)}, S_{k(x)}) = 0$ and the corresponding regularized action principle, it follows from the equation independent of the sources $k(x)$ that the divergences $\Gamma^{(1)}{}_{div}$ of the theory without $k(x)$ are, as usual, required to be BRST invariant. The terms linear in $k(x)$ then imply

$$\left(\left.\frac{\delta \Gamma^{(1)}_{k(x)div}}{\delta k^\Gamma(x)}\right|_{k(x)=0}, S \right) + (O_\Gamma(x), \Gamma^{(1)}{}_{div}) = 0\,. \tag{A.1}$$

The second term of this equation gives for the antifield dependent counterterms $\Gamma^{(1)}{}_{div} = K'$ above $(O_\Gamma(x), K') = (C^\alpha f^\Delta_\alpha \delta_\Delta) O_\Gamma(x)$, because we have choosen (for simplicity) $O_\Gamma(x)$ to be independent of the antifields. From (A.1), it then follows that $(C^\alpha f^\Delta_\alpha \delta_\Delta) O_\Gamma(x)$ must be given on-shell by a gauge transformation, involving the abelian ghosts alone, of polynomials $P_\Gamma(x)$. This follows by using the explicit form of the BRST differential, and after evaluation, putting to zero the antifields, and the non abelian ghosts. Hence, the counterterms K' are excluded a priori whenever it is possibe to construct $O_\Gamma(x)$'s for which the corresponding $P_\Gamma(x)$ do not exist so that (A.1) cannot be satisfied.

Remark. Because we use external sources instead of coupling constant, the example given here is not covered by the general analysis done in this paper. Note however that it is possible to generalize the antibracket map considered in [23, 24] to a mixed antibracket map from the tensor product of integrated times non integrated cohomology classes to non integrated cohomology classes as needed in this example. The invariant cohomological statement of which we have discussed a particular case above is: *the counterterms are restricted to belong to the kernel of the mixed antibracket map.* We use external sources in this example because the restrictions we get are stronger and the discussion is simplified: we need not worry about possible integrations by parts (in momentum space, this means that the restrictions we get are valid for all values of the external momentum and not only for zero external momentum).

This means that besides the arguments of [8, 21, 53], there exists an elegant cohomological mechanism to eliminate this type of antifield dependent counterterms.

In the concrete case of the standard model, the global symmetries δ_Δ correspond to lepton and baryon number conservation. There is only one abelian ghost C^α, the abelian gauge transformation of the matter fields being $\delta_{abelian} y = i\mathcal{Y} y$, where $\mathcal{Y} = \mathcal{Y}^i_j y^j \frac{\partial}{\partial y^i}$ is the hypercharge. As an example of O_Γ's we can take any three linearily independent operators out of the lepton number non conserving gauge invariant operators of dimension 5 in the matter fields given in eq. (20) of [54] (they can also be found in eq. (21.3.54) of [55]) and one baryon number non conserving operator out of the six dimension 6 gauge invariant operators given in eqs. (1)–(6) in [54, 56].

[12] The author thanks P.A. Grassi for suggesting the use of external sources in this example.

JHEP03(1999)010

Because these operators are build out of the undifferentiated matter fields alone, a sufficient condition for (A.1) to hold is the existence of $P'_\Gamma(x)$'s build out of the undifferentiated y^i such that

$$f^\Gamma n_\Gamma O_\Gamma = \mathcal{Y} P'_\Gamma\,, \tag{A.2}$$

(with no summation over Γ) where n_Γ is the lepton number of the O_Γ's for $\Gamma = 1, 2, 3$ and the baryon number for O_4. This follows by identifying the term in the abelian ghost and putting, in addition to the non abelian ghosts and the antifields, the derivatives of the abelian ghost, the derivatives of the matter fields and all the gauge fields to zero and using the fact that the equations of motion necessarily involve derivatives. Because the O_Γ's we have choosen are all of homogeneity 4 in the y^i and $\mathcal{Y}$ is of homogeneity 0, we can assume that the homogeneity of the P'_Γ's is also 4. By decomposing the space M_4 of monomials of homogenity 4 in the y^i into eigenspaces of the hermitian operator $\mathcal{Y}$ with definite eigenvalues $M_4 = M_4^0 + \oplus_{n\neq 0} M_4^n$, it follows that (A.2) has no nontrivial solutions. Indeed, decomposing $P'_\Gamma = P^{0\prime}_\Gamma + \sum_{n\neq 0} P^{n\prime}_\Gamma$, (A.2) reads $f^\Gamma n_\Gamma O_\Gamma = \sum_{n\neq 0} n P^{n\prime}_\Gamma$. Applying $\mathcal{Y}$ k times and using the fact that gauge invariance of O_Γ implies $\mathcal{Y} O_\Gamma = 0$, we get $\sum_{n\neq 0} n^k P^{n\prime}_\Gamma = 0$. We then can conclude that $P^{n\prime}_\Gamma = 0$ for $n \neq 0$, which implies $f^\Gamma = 0$.

As usual, this one loop reasoning can be extended recursively to higher orders, or alternatively, it can be discussed independently of the assumption that there exists an invariant regularization scheme in the context of algebraic renormalization.

References

[1] L.D. Faddeev and V.N. Popov, *Phys. Lett.* **B 25** (1967) 29.

[2] C. Becchi, A. Rouet and R. Stora, *Comm. Math. Phys.* **42** (1975) 127; *Ann. Phys. (NY)* **98** (1976) 287.

[3] I.V. Tyutin, *Gauge invariance in field theory and statistical mechanics*, Lebedev preprint FIAN, $n^0$39, 1975.

[4] J. Zinn-Justin, *Renormalisation of gauge theories* in *Trends in elementary particle theory*, *Lecture notes in Physics* **37**, Springer, Berlin 1975; *Quantum Field Theory and Critical Phenomena*, 2nd Edition, Clarendon Press, Oxford, 1993.

[5] J.A. Dixon, *Nucl. Phys.* **B 99** (1975) 420.

[6] H. Kluberg-Stern and J.B. Zuber, *Phys. Rev.* **D 12** (1975) 467; *Phys. Rev.* **D 12** (1975) 482; *Phys. Rev.* **D 12** (1975) 3159.

[7] G. Barnich and M. Henneaux, *Phys. Rev. Lett.* **72** (1994) 1588 [hep-th/9312206]; G. Barnich, F. Brandt and M. Henneaux, *Comm. Math. Phys.* **174** (1995) 93 [hep-th/9405194].

JHEP03(1999)010

[8] G. Bandelloni, C. Becchi, A. Blasi and R. Collina, *Ann. Inst. Henri Poincaré*, **A28** (1978) 225; *Ann. Inst. Henri Poincaré*, **A28** (1978) 255.

[9] I.A. Batalin and G.A. Vilkovisky, *Phys. Lett.* **B 102** (1981) 27; *Phys. Rev.* **D 28** (1983) 2567; *Phys. Rev.* **D 30** (1984) 508; *Nucl. Phys.* **B 234** (1984) 106; *J. Math. Phys.* **26** (1985) 172.

[10] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, Princeton, 1992.

[11] J. Gomis, J. Parìs and S. Samuel, *Phys. Rep.* **259** (1995) 1 [hep-th/9412228].

[12] A. Blasi and R. Collina, *Nucl. Phys.* **B 285** (1987) 204;
C. Becchi, A. Blasi, G. Bonneau, R. Collina and F. Delduc, *Comm. Math. Phys.* **120** (1988) 121.

[13] F. Brandt, M. Henneaux and A. Wilch, *Nucl. Phys.* **B 510** (1998) 640 [hep-th/9705007].

[14] B.L. Voronov and I.V. Tyutin, *Theor. Math. Phys* **50** (1982) 218; *Theor. Math. Phys* **52** (1982) 628.

[15] B.L. Voronov, P.M. Lavrov and I.V. Tyutin, *Sov. J. Nucl. Phys.* **36** (1982) 292.

[16] P.M. Lavrov and I.V. Tyutin, *Sov. J. Nucl. Phys.* **41** (1985) 1049.

[17] J.M.L. Fisch and M. Henneaux, *Comm. Math. Phys.* **128** (1990) 627.

[18] M. Henneaux, *Comm. Math. Phys.* **140** (1991) 1.

[19] G. Barnich, F. Brandt and M. Henneaux, *Comm. Math. Phys.* **174** (1995) 57 [hep-th/9405109].

[20] D. Anselmi, *Class. and Quant. Grav.* **11** (1994) 2181 [hep-th/9407023].

[21] J. Gomis and S. Weinberg, *Nucl. Phys.* **B 469** (1996) 473 [hep-th/9510087].

[22] S. Weinberg, *What is Quantum Field Theory, and What Did We Think It Is?*, hep-th/9702027.

[23] G. Barnich and M. Henneaux, *Phys. Lett.* **B 311** (1993) 123 [hep-th/9304057].

[24] G. Barnich, *Phys. Lett.* **B 419** (1998) 211 [hep-th/9710162].

[25] W. Troost, P. van Nieuwenhuizen and A. Van Proeyen, *Nucl. Phys.* **B 333** (1990) 727.

[26] P.S. Howe, U. Lindström and P.L. White, *Phys. Lett.* **B 246** (1990) 430.

[27] F. De Jonghe, J. Parìs and W.Troost, *Nucl. Phys.* **B 476** (1996) 559 [hep-th/9603012].

JHEP03(1999)010

[28] J. Parìs, *Nucl. Phys.* **B 450** (1995) 357 [hep-th/9502140];
J. Parìs and W.Troost, *Nucl. Phys.* **B 482** (1996) 373 [hep-th/9607215].

[29] V.S. Retakh, *J. Pure Appl. Algebra* **89** (1993) 217.

[30] G. t'Hooft and M. Veltman, *Nucl. Phys.* **B 44** (1972) 189.

[31] T. Marinucci and M. Tonin, *Nuovo Cim.* **31A** (1976) 381;
G. Costa, T. Marinucci, J. Julve and M. Tonin, *Nuovo Cim.* **38A** (1977) 373;
S. Aoyama and M. Tonin, *Nucl. Phys.* **B 179** (1981) 293.

[32] M. Tonin, *Nucl. Phys.* **29BC** *(Proc. Suppl.)* (1992) 137.

[33] P. Breitenlohner and D. Maison, *Comm. Math. Phys.* **52** (1977) 11; *Comm. Math. Phys.* **52** (1977) 39; *Comm. Math. Phys.* **52** (1977) 55.

[34] G. Bonneau, *Nucl. Phys.* **B 167** (1980) 261; *Nucl. Phys.* **B 171** (1980) 477; *Nucl. Phys.* **B 177** (1981) 523; *Nucl. Phys.* **B 221** (1983) 178;
G. Bonneau and F. Delduc, *Nucl. Phys.* **B 266** (1986) 536.

[35] O. Piguet and A. Rouet, *Phys. Rep.* **76** (1981) 1.

[36] O. Piguet and S.P. Sorella, *Algebraic Renormalization*, *Lecture Notes in Physics n 28*, Springer Verlag, Berlin, Heidelberg, 1995.

[37] G. Bonneau, *Int. J. Mod. Phys.* **A 5** (1990) 3831.

[38] E.S. Fradkin and G.A. Vilkovisky, *Phys. Lett.* **B 55** (1975) 224; *Quantization of relativistic systems with constraints: equivalence of canonical and covariant formalism in quantum theory of gravitational field*, CERN report TH2332, 1977;
I.A. Batalin and G.A. Vilkovisky, *Phys. Lett.* **B 69** (1977) 309;
E.S. Fradkin and T.E. Fradkina, *Phys. Lett.* **B 72** (1978) 343.

[39] E. Witten *Mod. Phys. Lett.* **A 5** (1990) 487.

[40] M. Henneaux, *Nucl. Phys.* **18A** *(Proc. Suppl.)* (1990) 47.

[41] G. Hirsch, *Bull. Soc. Math. Belg.* **6** (1953) 79;
J.D. Stasheff, *Trans. Am. Math. Soc.* **108** (1963) 215; *Trans. Am. Math. Soc.* **108** (1963) 293;
V.K.A.M. Gugenheim, *J. Pure Appl. Alg.* **25** (1982) 197;
V.K.A.M. Gugenheim and J.D. Stasheff, *Bull. Soc. Math. Belg.* **38** (1986) 237.

[42] S. Weinberg, *Ultraviolet divergences in quantum gravity* in *General Relativity*, eds. S.W. Hawking and W. Israel, Cambridge University Press, 1979.

[43] J.H. Lowenstein, *Phys. Rev.* **D 4** (1971) 2281; *Comm. Math. Phys.* **24** (1971) 1;
Y.M.P. Lam, *Phys. Rev.* **D 6** (1972) 2145; *Phys. Rev.* **D 7** (1973) 2943;
T.E. Clark and J.H. Lowenstein, *Nucl. Phys.* **B 113** (1976) 109.

JHEP03(1999)010

[44] G. Barnich, *A general non renormalization theorem in the extended antifield formalism*, *J. High Energy Phys.* **12** (1998) 003 [hep-th/9805030].

[45] P.L. White, *Phys. Lett.* **B 284** (1992) 55.

[46] O. Piguet and K. Sibold, *Nucl. Phys.* **B 253** (1985) 517.

[47] J.M.L. Fisch, *A note on the extended BRST approach to the gauge (in)dependence in the renormalization of gauge theories*, preprint ULB-TH2/89-09; *On the Batalin-Vilkovisky Antibracket-Antifield BRST Formalism and its Applications*, preprint ULB-TH2-90-01.

[48] G. Barnich, F. Brandt and M. Henneaux, *Phys. Rev.* **D 51** (1995) R1435 [hep-th/9409104]; *Nucl. Phys.* **B 455** (1995) 357 [hep-th/9505173].

[49] F. Brandt, W. Troost, A. Van Proeyen, *Nucl. Phys.* **B 464** (1996) 353 [hep-th/9509035].

[50] M. Henneaux and B. Knaepen, *Phys. Rev.* **D 56** (1997) 6076 [hep-th/9706119]; M. Henneaux, B. Knaepen and C. Schomblond, *Comm. Math. Phys.* **186** (1997) 137 [hep-th/9606181]; *Lett. Math. Phys.* **42** (1997) 337 [hep-th/9702042].

[51] F. Brandt, *Ann. Phys. (NY)* **259** (1997) 253 [hep-th/9609192].

[52] F. Brandt, J. Gomis and J. Simón, *Nucl. Phys.* **B 523** (1998) 623 [hep-th/9712125].

[53] P.A. Grassi, *Nucl. Phys.* **B 537** (1999) 527 [hep-th/9804013].

[54] S. Weinberg, *Phys. Rev. Lett.* **43** (1979) 1566.

[55] S. Weinberg, *The Quantum Theory of Fields II*, Cambridge University Press, 1996.

[56] F. Wilczek and A. Zee, *Phys. Rev. Lett.* **43** (1979) 1571.

JHEP03(1999)010